An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature

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Abstract

We formulate natural conformally invariant conditions on a 4-manifold for the existence of a metric whose Schouten tensor satisfies a quadratic inequality. This inequality implies that the eigenvalues of the Ricci tensor are positively pinched.

Introduction

Conformal geometry in two dimensions is distinguished by its relationship to complex analysis. In higher dimensions the landscape becomes more complicated, and in the absence of some special structure (e.g., Kähler) even an extensive knowledge of the theory of Riemann surfaces is no longer a reliable guide.

Our setting in this paper is four dimensions, and one of our goals is to propose a point of view which emphasizes certain parallels between conformal geometry in two and four dimensions. To illustrate this, let us begin by recalling the Gauss-Bonnet formula for compact surfaces:

$$(0.1) \qquad 2\pi\chi(M^2) = \int_{M^2} K \, d\mu$$

where $d\mu$ is the area element, and K is the Gauss curvature of the surface. In four dimensions the Chern-Gauss-Bonnet integrand is a quadratic polynomial in the curvature; nevertheless there is a strong analogy with (0.1).

To see this, let (M^4, g) be a compact Riemannian four-manifold, and let W, Ric, and R denote respectively the Weyl curvature tensor, Ricci tensor, and scalar curvature of g. To express the Chern-Gauss-Bonnet formula it will be helpful to introduce the elementary symmetric functions $\sigma_k : \mathbb{R}^n \to \mathbb{R}$, $1 \leq k \leq n$. Given a section A of the bundle of symmetric two-tensors, we

can use the metric to raise an index and view A as a tensor of type (1,1), or equivalently as a section of $\mathrm{End}(TM^4)$. Under this identification, $\sigma_k(A)$ means σ_k applied to the eigenvalues of A.

In particular, let $A = \mathrm{Ric} - \frac{1}{6}Rg$ denote the Schouten tensor. With the notation described above the Chern-Gauss-Bonnet formula can be written

$$(0.2) \qquad 8\pi^2\chi(M^4) = \frac{1}{4}\int |W|^2\,dv + \int \sigma_2(A)dv,$$

where dv denotes the volume form. When we compare (0.1) and (0.2), a certain parallel emerges between the Gauss curvature of a surface and the quantity $\sigma_2(A)$ of a four-manifold, despite the presence of the Weyl curvature term (0.2). Indeed, this term actually strengthens the analogy: recall that the Weyl tensor measures whether the four-manifold is locally conformally flat (LCF). But every surface is LCF, so the obstruction is vacuous and the corresponding term is absent in (0.1) (or if one prefers, it is zero).

A further parallel between $\int K\,d\mu$ and $\int \sigma_2(A)dv$ is that both are conformally invariant. This is obvious for the Gauss curvature; for $\sigma_2(A)$ it follows from (0.2) and the conformal invariance of $\int |W|^2dv$.

This is actually a special case of a more general phenomenon. Let (M^{2k}, g) be a compact, LCF Riemannian manifold of dimension $n = 2k$. If we define $A = \mathrm{Ric} - \frac{1}{2(n-1)}Rg = \mathrm{Ric} - \frac{1}{2(2k-1)}Rg$, then the integral

$$\int \sigma_k(A)\,dv$$

is conformally invariant (see [V-1]). Moreover,

$$\chi(M^{2k}) = c_k \int \sigma_k(A)\,dv.$$

Returning to four dimensions, we have a further parallel between the integrals in (0.1) and (0.2): if $\int_{M^2} K dv > 0$ then M^2 has genus zero; on the other hand, if (M^4, g) has positive scalar curvature and $\int \sigma_2(A)dv > 0$, then the first Betti number $b_1(M^4) = 0$ (see [G-1]).

The foregoing observation provided the motivation for the main result of the present paper. To understand how, recall the classical vanishing theorem of Bochner: a compact Riemannian manifold of positive Ricci curvature has $b_1 = 0$. The assumption on the Ricci curvature is natural in light of the famous Weitzenbock formula for harmonic one-forms ω:

$$\frac{1}{2}\Delta |\omega|^2 = |\nabla\omega|^2 + \mathrm{Ric}(\omega, \omega).$$

It is a little surprising, therefore, that the positivity of conformal invariants like $\int \sigma_2(A)dv$ and the Yamabe invariant should also imply the vanishing of b_1.

So one is lead to conjecture: Suppose (M^4, g_0) is a compact four-manifold with $\int \sigma_2(A_0) dv_0 > 0$ and Yamabe invariant $Y(g_0) > 0$. Is there a conformal metric $g = e^{2w} g_0$ with strictly positive Ricci curvature?

To understand our approach to this conjecture, it will be illuminating to point out a relationship between the Ricci tensor and $\sigma_2(A)$. Suppose $g = e^{2w} g_0$ with $\sigma_2(A_g) > 0$. From the positivity of the Yamabe invariant of g_0, it is not difficult to conclude that the scalar curvature R of g must also be positive (cf. Lemma 1.1). Moreover, the Ricci curvature of g satisfies the inequality

$$\text{Ric} \geqslant \frac{3\sigma_2(A)}{R} g.$$

In particular, Ric is positive.[1]

The preceding tells us that we can solve our conjecture in the affirmative by constructing a conformal metric $g = e^{2w} g_0$ with $\sigma_2(A_g) > 0$, assuming only that $\int \sigma_2(A_0) dv_0 > 0$ and $Y(g_0) > 0$.

We remark that the positivity of $\sigma_2(A)$ is a much stronger condition than positive Ricci curvature. Indeed, if we define $S = -\text{Ric} + \frac{1}{2} Rg$ (in general relativity this is the gravitational tensor) then S also satisfies the inequality

$$S \geqslant \frac{3\sigma_2(A)}{R} g$$

(see Lemma 1.2). Thus, $\sigma_2(A) > 0$ imposes a pinching condition on the Ricci curvature; it implies that each eigenvalue of Ric is positive, but less than the sum of the other three. Moreover, if (M^4, g) is oriented with $\int \sigma_2(A) dv > 0$, then the Euler characteristic $\chi(M^4)$ and signature $\tau(M^4)$ must satisfy the inequality

(0.3) $$\chi(M^4) > \frac{3}{2} |\tau(M^4)|$$

(see [G-2]). However, there are many examples of four-manifolds with positive Ricci curvature which violate (0.3) (see [ShYa]). This is discussed in more detail Section 8.

The main analytic difficulty arising in the study of $\sigma_2(A)$ is the following. If we write $g = e^{2w} g_0$, then the tensor A of g is related to the tensor A_0 of g_0 by the identity

(0.4) $$A = A_0 - 2\nabla_0^2 w + 2dw \otimes dw - |dw|^2 g_0,$$

where ∇_0^2 denotes the Hessian with respect to g_0. In light of (0.4), the equation under consideration is

(0.5) $$\sigma_2(A_0 - 2\nabla_0^2 w + 2dw \otimes dw - |dw|^2 g_0) = f > 0.$$

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[1] In dimensions greater than four, $\sigma_2(A) > 0$ implies that the scalar curvature has a sign, but not the Ricci curvature. In three dimensions $\sigma_2(A) > 0$ actually implies a sign on the *sectional* curvature; see [GV].

This is an example of a fully nonlinear equation of Monge-Ampère type. Introducing local coordinates, one can view (0.5) as an equation of the form

$$(0.6) \qquad F[\partial_i\partial_j w, \partial_k w, w, x] = f$$

where

$$F : \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},$$

given by $F(r_{ij}, v_k, s, x) = f$. Then (0.6) is *elliptic* at a solution w if the matrix $\frac{\partial F}{\partial r_{ij}}$ is positive definite. In the case of (0.5), this will hold provided $f > 0$ (see Proposition 1.5). The classical techniques for analyzing such equations usually begin by assuming that one has some kind of approximate solution (sub-, super-, viscosity, etc.) which is elliptic. As we shall see, equation (0.5) is elliptic at w if and only if the tensor $S = -\mathrm{Ric} + \frac{1}{2}Rg$ is positive definite for the metric $g = e^{2w}g_0$. That is, we are confronted with the difficulty of solving (0.5) without knowing *a priori* that our linearized operator is elliptic.

Despite these difficulties, we are able to resolve the conjecture in the affirmative:

THEOREM A. *Let* (M^4, g_0) *be a compact four-manifold satisfying* (i) $\int \sigma_2(A_0)dv_0 > 0$ *and* (ii) $Y(g_0) > 0$. *Then there is a conformal metric* $g = e^{2w}g_0$ *with* $\sigma_2(A_g) > 0$.

COROLLARY B. *Under the assumptions of Theorem* A, *there is a conformal metric* $g = e^{2w}g_0$ *with*

(i) $\mathrm{Ric} > 0$,

(ii) $S = -\mathrm{Ric} + \frac{1}{2}Rg > 0$.

It is natural to ask for which manifolds the assumptions of the theorem are likely to hold. This question is addressed in Section 8, where we use Freedman's [F] work to give a list of the simply connected candidates (up to homeomorphism). In addition, we construct some explicit examples.

Assuming M^4 is orientable, by combining the Chern-Gauss-Bonnet formula

$$8\pi^2 \chi(M^4) = \frac{1}{4}\int |W|^2 \, dv + \int \sigma_2(A) \, dv$$

with the signature formula

$$12\pi^2 \tau(M^4) = \frac{1}{4}\int (|W^+|^2 - |W^-|^2) \, dv$$

we obtain

$$2\pi^2(2\chi(M^4) + 3\tau(M^4)) = \frac{1}{4}\int |W^+|^2 \, dv + \frac{1}{2}\int \sigma_2(A) \, dv.$$

Therefore, an equivalent formulation of Theorem A is

COROLLARY C.   *If (M^4, g_0) is an oriented compact four-manifold satisfying $Y(g_0) > 0$ and*

$$\frac{1}{4} \int |W_0^+|^2 \, dv_0 \; < \; 2\pi^2 (2\chi(M^4) + 3\tau(M^4)),$$

then there is a conformal metric $g = e^{2w} g_0$ with $\sigma_2(A_g) > 0$ (hence with $\mathrm{Ric} > 0$, $S > 0$).

The problem of conformally deforming a metric with $\sigma_2(A) > 0$ to one with $\sigma_2(A) \equiv$ constant is addressed - but not resolved - in [V-2], where degree-theoretic arguments are used. What is lacking are L^∞-estimates for solutions of (0.5). In a subsequent paper we present an alternative approach, including *a priori* L^∞-bounds for solutions of (0.5) on manifolds that are not conformally equivalent to the round four-sphere [CGY-2].

We conclude the introduction with some remarks about the structure of the proof and the organization of the paper.

To overcome the lack of ellipticity for the linearized problem, we will regularize our equation by a geometrically natural fourth order term. The regularized equation actually arises in spectral theory, in the context of the zeta functional determinant of a conformally invariant operator (see Section 2). More precisely, our regularized equation is

$$(0.7) \qquad\qquad \sigma_2(A) \; = \; \frac{\delta}{4} \Delta R \; - \; 2\gamma_1 \, |\eta|^2$$

where ΔR is the Laplacian of the scalar curvature, $\delta > 0$ is small, $\gamma_1 < 0$ is a fixed constant, and η is a nowhere-vanishing section of the bundle of symmetric two-tensors. For each sufficiently small $\delta > 0$, we are able to show that (0.7) admits a smooth solution with positive scalar curvature (see Section 4).

The next (and most involved) step is obtaining *a priori* estimates for solutions of (0.7) that are independent of δ. This is accomplished in Sections 3,5, and 6 - at least up to a point. There seem to be technical obstructions preventing us from establishing anything beyond $C^{1,\alpha}$-estimates. But these estimates are adequate to prove that the regularizing term $\delta\Delta R$ in (0.7) is approaching zero in (roughly) an L^2-sense as $\delta \to 0$.

The final step of the proof is an application of heat equation techniques. Using the Yamabe flow, we show that solutions to (0.7) can be perturbed to give metrics with $\sigma_2(A) > 0$, once δ is sufficiently small. This is explained in Section 7.

1. Background

In this section we establish our notation, sketch some background material, and prove some technical lemmas.

a. *The curvature of four-manifolds.* To begin, let (M^4, g) be a compact four-manifold. The curvature tensor will be denoted Rm, and usually viewed as a $(0, 4)$-tensor. We let W, Ric, and R denote respectively the Weyl curvature, Ricci curvature, and scalar curvature of g. There are various ways to decompose the curvature tensor under the action of the orthogonal group, depending on the basis one chooses. If $E = \text{Ric} - \frac{1}{4}Rg$ denotes the trace-free Ricci tensor, then

$$(1.1) \qquad Rm = W + \frac{1}{2}E \owedge g + \frac{1}{24}Rg \owedge g$$

where $\owedge$ is the Kulkarni-Nomizu product (see [Be, 1.110]). Alternatively, if $A = \text{Ric} - \frac{1}{6}Rg$, then we have the somewhat simpler decomposition

$$(1.2) \qquad Rm = W + \frac{1}{2}A \owedge g.$$

In conformal geometry there are certain computational advantages to working with (1.2) instead of (1.1).

If $\chi(M^4)$ denotes the Euler characteristic of M^4, then by the Chern-Gauss-Bonnet formula

$$(1.3) \qquad \chi(M^4) = \int \text{Pf}f(Rm)$$

where $\text{Pf}f(Rm)$ denotes the Pfaffian of the curvature (now viewed as a matrix of Lie algebra-valued two-forms). Using the basis in (1.2), we can also express (1.3) as

$$(1.4) \qquad \chi(M^4) = \frac{1}{8\pi^2} \int \left(\frac{1}{4}|W|^2 + \sigma_2(A) \right) dv.$$

If M^4 is oriented, let $* : \Omega^p(M^4) \to \Omega^{4-p}(M^4)$ denote the Hodge operator. Then we have the splitting $\Omega^2(M^4) = \Omega_+^2(M^4) \oplus \Omega_-^2(M^4)$ into the sub-bundles of self-dual and anti-self-dual two-forms. This splitting induces a decomposition of the Weyl curvature into $W^\pm : \Omega_\pm^2(M^4) \to \Omega_\pm^2(M^4)$, viewed as as bundle endomorphism. Combining the signature formula

$$12\pi^2 \tau(M^4) = \frac{1}{4} \int \left(|W^+|^2 - |W^-|^2 \right)$$

with (1.4) we obtain

$$(1.5) \qquad 2\pi^2(2\chi(M^4) + 3\tau(M^4)) = \frac{1}{4}\int |W^+|^2 + \frac{1}{2}\sigma_2(A).$$

It is clear from (1.4) and (1.5) that the positivity of $\sigma_2(A)$ implies global topological information. But it also implies local geometric information, as the following lemmas show.

LEMMA 1.1 (see [V-1, Lemma 23]). $R^2 \geq 24\sigma_2(A)$ *with equality if and only if* $E = 0$. *In particular, if* $\sigma_2(A) > 0$ *on* M^4 *then either* $R > 0$ *or* $R < 0$ *on* M^4.

Proof. This is immediate, since

$$\sigma_2(A) = -\frac{1}{2}|E|^2 + \frac{1}{24}R^2 \leq \frac{1}{24}R^2. \qquad \square$$

LEMMA 1.2. *Let* $P \in M^4$ *and* $X \in T_P M^4$ *be a tangent vector at* P. *If the scalar curvature* R *of* g *is positive at* P, *then*

$$(1.6) \qquad S(X,X) \quad = \quad -\operatorname{Ric}(X,X) + \frac{R}{2}g(X,X)$$

$$\geq \quad \frac{3}{R}\sigma_2(A)\,g(X,X)$$

$$(1.7) \qquad \operatorname{Ric}(X,X) \quad \geq \quad \frac{3}{R}\sigma_2(A)\,g(X,X).$$

Proof. To simplify notation we often denote $g(X,X) = |X|^2 = \langle X, X \rangle$. In terms of the trace-free Ricci tensor,

$$(1.8) \qquad S = -E + \frac{1}{4}Rg,$$

so that

$$S(X,X) = -E(X,X) + \frac{1}{4}Rg(X,X).$$

Since E is trace-free, we have the sharp inequality $|E(X,X)| \leq \frac{\sqrt{3}}{2}|E|\,|X|^2$ (see [SW, p. 234]). Thus

$$S(X,X) \quad \geq \quad -\frac{\sqrt{3}}{2}|E|\,|X|^2 + \frac{1}{4}R|X|^2$$

$$= \quad -2\left(|E|\sqrt{\frac{3}{2R}}\right)\left(\sqrt{\frac{R}{8}}\right)|X|^2 + \frac{1}{4}R|X|^2$$

$$\geq \quad -\left(|E|\sqrt{\frac{3}{2R}}\right)^2|X|^2 - \left(\sqrt{\frac{R}{8}}\right)^2|X|^2 + \frac{1}{4}R|X|^2$$

$$= \left(-\frac{3}{2}\frac{|E|^2}{R} + \frac{1}{8}R\right)|X|^2$$

$$= \frac{3}{R}\sigma_2(A)|X|^2.$$

The proof of (1.7) is essentially the same. We begin with

(1.9) $$\mathrm{Ric} = E + \frac{1}{4}Rg.$$

Then

$$\mathrm{Ric}(X,X) \geq -\frac{\sqrt{3}}{2}|E|\,|X|^2 + \frac{1}{4}R|X|^2,$$

and we can argue as before.   □

Remark. It is worthwhile comparing (1.8) and (1.9). Recall that every symmetric two-tensor can be decomposed into a trace-free part and a pure trace part. The identities (1.8) and (1.9) show that S and Ric have the same pure trace component under this decomposition, but their trace-free components differ by a sign.

Arguing exactly as in the proof of Lemma 1.2 we have

LEMMA 1.3.   *Let $P \in M^4$ and $X \in T_P M^4$. If $R < 0$ at P then*

$$S(X,X) \leq \frac{3}{R}\sigma_2(A)\,g(X,X),$$

$$\mathrm{Ric}(X,X) \leq \frac{3}{R}\sigma_2(A)\,g(X,X).$$

Combining the preceding lemmas we conclude:

COROLLARY 1.4.   *If $\sigma_2(A) > 0$ on M^4 then either $S > 0$ and $\mathrm{Ric} > 0$ on M^4, or $S < 0$ and $\mathrm{Ric} < 0$ on M^4, depending on the sign of the scalar curvature (which is necessarily constant by Lemma 1.1).*

b. *Conformal changes of metric.* Now denote our four-manifold by (M^4, g_0). We will usually write conformal metrics in the form $g = e^{2w}g_0$. Also, metric-dependent quantities which have 0 as a subscript or superscript are understood to be with respect to g_0, while those without are with respect to g. For example, $\nabla_0^2\varphi$ denotes the Hessian of φ with respect to g_0 and $\Delta_0\varphi = tr_{g_0}\nabla_0^2\varphi$ the Laplacian; while $\nabla^2\varphi$ and $\Delta\varphi = \mathrm{tr}_g\nabla^2\varphi$ denote the Hessian and Laplacian with respect to g.

Of basic importance are the transformation laws for the various components of the curvature tensor under a conformal change of metric:

(1.10) $R = e^{-2w}(R_0 - 6\Delta_0 w - 6|\nabla_0 w|^2),$

(1.11) $\mathrm{Ric} = \mathrm{Ric}_0 - 2\nabla_0^2 w - \Delta_0 w g_0 + 2dw \otimes dw - 2|\nabla_0 w|^2 g_0,$

$$(1.12) \quad A \;=\; A_0 - 2\nabla_0^2 w + 2dw \otimes dw - |\nabla_0 w|^2 g_0,$$

$$(1.13) \quad S \;=\; S_0 + 2\nabla_0^2 w - 2\Delta_0 w g_0 - 2dw \otimes dw - |\nabla_0 w|^2 g_0.$$

It will often be useful to rewrite the above identities so that the covariant derivatives are taken with respect to g instead of g_0. In this case,

$$(1.14) \quad R \;=\; R_0 e^{-2w} - 6\Delta w + 6|\nabla w|^2,$$

$$(1.15) \quad \mathrm{Ric} \;=\; \mathrm{Ric}_0 - 2\nabla^2 w - \Delta w g - 2dw \otimes dw + 2|\nabla w|^2 g,$$

$$(1.16) \quad A \;=\; A_0 - 2\nabla^2 w - 2dw \otimes dw + |\nabla w|^2 g,$$

$$(1.17) \quad S \;=\; S_0 + 2\nabla^2 w - 2\Delta w g + 2dw \otimes dw + |\nabla w|^2 g.$$

The Bach tensor plays a prominent role in our analysis. It is defined by (see [De])

$$B_{ij} \;=\; \nabla^k \nabla^\ell W_{kij\ell} + \frac{1}{2} R^{k\ell} W_{kij\ell}.$$

Using the Bianchi identities, we can rewrite this as

$$(1.18) \quad \begin{aligned} B_{ij} \;=\; & -\frac{1}{2}\Delta E_{ij} + \frac{1}{6}\nabla_i \nabla_j R - \frac{1}{24}\Delta R g_{ij} - E^{k\ell} W_{ikj\ell} \\ & + E_i^k E_{jk} - \frac{1}{4}|E|^2 g_{ij} + \frac{1}{6} R E_{ij} \end{aligned}$$

where $\Delta E_{ij} = g^{k\ell} \nabla_k \nabla_\ell E_{ij}$. Although it has several interesting properties, for our purposes the most important feature of the Bach tensor is its conformal invariance: if $g = e^{2w} g_0$, then

$$(1.19) \qquad\qquad\qquad B \;=\; e^{-2w} B_0.$$

c. *Equations of Monge-Ampère type.* Since our eventual goal is to produce conformal metrics with $\sigma_2(A) > 0$, it will be helpful to provide some background for the analytic aspects of the problem. If we fix a background metric g_0, then by (1.12) we are attempting to solve the equation

$$(1.20) \quad \sigma_2(A_0 - 2\nabla_0^2 w + 2dw \otimes dw - |\nabla_0 w|^2 g_0) \;=\; f$$

for some $f > 0$. This is an example of a fully nonlinear equation of Monge-Ampère type (see [CNS-1], [CNS-2], [CKNS]). Many of the relevant properties of (1.20) are summarized by the following result:

PROPOSITION 1.5. *The equation* (1.20) *is elliptic at a solution w if $f > 0$. The linearized operator*

$$L[\varphi] = \frac{\partial F}{\partial r_{ij}}(\nabla_0^2 \varphi)_{ij}$$

(*see the introduction*) *is given by*

$$(1.21) \qquad\qquad\qquad L[\varphi] \;=\; -2 S^{ij}\, \nabla_i^0 \nabla_j^0 \varphi,$$

where $S^{ij} = e^{-4w}(g_0)^{ik}(g_0)^{j\ell}S_{k\ell}$, *and*

$$S_{k\ell} = S_{k\ell}^0 + 2\nabla_k^0\nabla_\ell^0 w - 2(\Delta_0 w)(g_0)_{k\ell}$$
$$- 2\nabla_k^0 w\,\nabla_\ell^0 w - |\nabla_0 w|^2 (g_0)_{k\ell}$$

is given by (1.13). *If the scalar curvature* R *of* $g = e^{2w}g_0$ *is positive, then the ellipticity constants of* L *satisfy*

(1.22) $$\frac{1}{2}Rf|\xi|^2 \geq S_{ij}\xi^i\xi^j \geq \frac{3}{R}f\,|\xi|^2.$$

A proof of Proposition 1.5 can be found in [V-1]. We only remark that the estimates (1.22) follow from Lemma 1.2.

2. The functional determinant

Let (M^4, g_0) be a compact four-manifold. A metrically defined differential operator L is said to be conformally covariant of bidegree (a, b) if under the conformal change of metric $g = e^{2w}g_0$,

(2.1) $$L_g(\varphi) = e^{-bw}L_0(e^{aw}\varphi).$$

In [BO] an explicit formula for $F[w] = \log(\det L_g/\det L_0)$ is computed, which may be expressed as

(2.2) $$F[w] = \gamma_1 \mathrm{I}[w] + \gamma_2\,\mathrm{II}\,[w] + \gamma_3\,\mathrm{III}\,[w]$$

where $\gamma_i = \gamma_i(L)$ are constants and

(2.3) $$\mathrm{I}\,[w] = \int 4|W_0|^2 w\,dv_0 - \left(\int |W_0|^2 dv_0\right)\log \fint e^{4w} dv_0,$$

$$\mathrm{II}\,[w] = \int w P_0 w\,dv_0 + \int 4Q_0 w\,dv_0 - \left(\int Q_0\,dv_0\right)\log \fint e^{4w}\,dv_0,$$

$$\mathrm{III}\,[w] = 12\left(Y[w] - \frac{1}{3}\int \Delta_0 R_0 w\,dv_0\right),$$

$$Y[w] = \int \left(\Delta_0 w + |\nabla_0 w|^2\right)^2 dv_0 - \frac{1}{3}\int R_0 |\nabla_0 w|^2\,dv_0.$$

Here P denotes the *Paneitz* operator [P]:

$$P = (\Delta)^2 + d^*\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)d,$$

where d is the exterior derivative, d^* is the adjoint of d, and Q is the fourth order curvature invariant:

$$Q = \frac{1}{12}\left(-\Delta R + \frac{1}{4}R^2 - 3|E|^2\right).$$

Thus

$$Q = \frac{1}{2}\sigma_2(A) + \frac{1}{12}(-\Delta R).$$

Before we discuss the existence theory some remarks are in order, explaining the significance of these formulas. First, if we consider the functional II alone, then critical points satisfy

$$P_0 w + 2Q_0 = 2 \left(\int Q_0 \, dv_0 \right) e^{4w}.$$

In general, if $g = e^{2w} g_0$ is a conformal change of metric, then the quantity Q transforms according to the formula

$$P_0 w + 2Q_0 = 2Q e^{4w}$$

where $Q = Q(e^{2w} g_0)$. We therefore conclude that critical points of II are precisely those metrics which satisfy $Q \equiv$ constant.

To understand III, it is helpful to rewrite it. Let R and dv denote the scalar curvature and volume form of the metric $g = e^{2w} g_0$; then

$$\mathrm{III}\,[w] = \frac{1}{3} \left[\int R^2 \, dv - \int R_0^2 \, dv_0 \right].$$

From this expression it is easy to see that critical points of III satisfy $\Delta R \equiv$ constant. Since M^4 is compact, this implies that R is constant. Thus III is the quadratic version of the Yamabe functional.

In part, the interest of the functional determinant resides in the fact that it is a natural Lagrangian arising in spectral theory whose Euler equation combines these geometrically natural "sub-functionals."

In order to state the relevant existence result of [CY-1] we need to define further the conformal invariant

$$(2.4) \qquad \kappa_d = \gamma_1 \int |W_0|^2 \, dv_0 + \gamma_2 \int Q_0 \, dv_0.$$

THEOREM 2.1 ([CY-1, Th. 1.1]).   *Let (M, g_0) be a compact four-manifold. If $\gamma_2, \gamma_3 > 0$ and $\kappa_d < 8\gamma_2\pi^2$, then $\inf F(w)$ is attained by some function $w \in W^{2,2}$ and the metric $g = e^{2w} g_0$ satisfies*

$$(2.5) \qquad \gamma_1 |W|^2 + \gamma_2 Q - \gamma_3 \Delta R = \kappa_d \, \mathrm{vol}(g)^{-1}.$$

Furthermore, g is smooth ([CGY-1]).

Remark. We warn the reader that the notation of [CY-1] is different: the signs of γ_i are reversed. What is crucial is that γ_2 and γ_3 have the same sign. If they have opposite signs then the existence theory of [CY-1] is not applicable. There are examples arising in applications in other contexts in which γ_2 and γ_3 have opposite signs; see [Br].

It will suit our purposes to modify slightly the functional studied in [CY-1] and [CY-2]. To describe our variant, we begin by pointing out that the functional I in (2.2) does not involve any derivatives of the conformal factor w. In addition, I introduces the term $\gamma_1 |W|^2$ into the Euler equation (2.5)

— or more precisely the term $\gamma_1 e^{-4w}|W_0|^2$. In fact, any geometric quantity which transforms in the same manner as the Weyl curvature under a conformal change of metric behaves similarly. Let us illustrate this with a specific example.

Let $S_2(M^4) = \Gamma(\mathrm{Sym}(T^*M^4 \otimes T^*M^4))$ denote sections of the bundle of symmetric $(0,2)$-tensors on M^4. Then $T^*M^4 \otimes T^*M^4$ inherits a bundle metric in the usual way from TM^4. Moreover, if $g = e^{2w}g_0$ and $\eta \in S_2(M^4)$ then $|\eta|^2_g = e^{-4w}|\eta|^2_0$. In particular, this example enjoys the same conformal scaling properties as the norm of the Weyl curvature:

$$
\begin{aligned}
|W|^2 &= |W|^2_g \\
&= e^{-4w}|W_0|^2_{g_0} \\
&= e^{-4w}|W_0|^2.
\end{aligned}
$$

Analogous to (2.2) we can introduce the functionals

$$
(2.6) \qquad I[w] = \int 4|\eta|^2_0\, w\, dv_0 - \left(\int |\eta|^2_0 dv_0\right) \log \fint e^{4w} dv_0,
$$

$$
(2.7) \qquad F[w] = \gamma_1 I[w] + \gamma_2 II[w] + \gamma_3 III[w],
$$

where II and III are defined as in (2.3). We then have the corresponding existence result:

COROLLARY 2.2.   *Let (M^4, g_0) be a compact four-manifold and $\eta \in S_2(M^4)$. If $\gamma_2, \gamma_3 > 0$ and $\kappa_d \equiv \gamma_1 \int |\eta|^2_0 dv_0 + \gamma_2 \int Q_0\, dv_0 < 8\gamma_2 \pi^2$, then $\inf F$ is attained by some function $w \in W^{2,2}$. The metric $g = e^{2w}g_0$ satisfies the Euler equation*

$$
(2.8) \qquad \gamma_1|\eta|^2 + \gamma_2 Q - \gamma_3 \Delta R = \kappa_d vol(g)^{-1}
$$

and is moreover smooth.

The proof of Corollary 2.2 is identical in its details to the proof of Theorem 2.1, and will therefore be omitted.

What is the purpose of modifying our functional in this way? It is difficult to give a complete answer to this question in advance of the description of our regularized problem. Eventually, though, we will construct a conformal metric with the property that $\sigma_2(A)$ is bounded below by a positive constant times $|\eta|^2$, plus error terms. Now it is easy to choose a section of $S_2(M^4)$ which is nowhere vanishing on M^4, and this means that $\sigma_2(A)$ is positive. But if the minimum of $\sigma_2(A)$ depended instead on $|W|^2$, then we could not in general rule out the possibility that $|W|^2$ (and hence $\sigma_2(A)$) vanishes somewhere.

To this end, let us fix once and for all a section $\eta \in S_2(M^4)$ which is nowhere vanishing (this is always possible: just let η be an arbitrary Riemannian metric on M^4). Let $\delta \in (0, 1]$, and set

$$
\begin{aligned}
(2.9) \qquad \gamma_1 &= -\int Q_0 dv_0 \Big/ \int |\eta|_0^2 \, dv_0 \\
&= -\frac{1}{2} \int \sigma_2(A_0) \, dv_0 \Big/ \int |\eta|_0^2 \, dv_0, \\
\gamma_2 &= 1, \\
\gamma_3 &= \frac{1}{24}(3\delta - 2).
\end{aligned}
$$

Remark. When $\delta < \frac{2}{3}$, according to (2.9), $\gamma_3 < 0$. Thus we are considering values of γ_3 for which the existence result of Corollary 2.2 is inapplicable. This will be explained below.

Notice that if $\int \sigma_2(A_0) dv_0 > 0$ then $\gamma_1 < 0$. With this choice of $(\gamma_1, \gamma_2, \gamma_3)$,

$$
(2.10) \qquad \kappa_d = \gamma_1 \int |\eta|_0^2 \, dv_0 + \int Q_0 dv_0 = 0.
$$

To write down the corresponding functional, let us introduce the quantity

$$
\begin{aligned}
(2.11) \qquad U_0^\delta &= U^\delta(g_0) \\
&= \gamma_1 |\eta|_0^2 + Q_0 - \frac{1}{24}(3\delta - 2) \Delta_0 R_0.
\end{aligned}
$$

Then according to (2.3), (2.6), (2.7), and (2.9),

$$
\begin{aligned}
(2.12) \qquad F[w] = F_\delta[w] &= \gamma_1 I[w] + II[w] + \frac{1}{24}(3\delta - 2) III[w] \\
&= \int 4 U_0^\delta \, w \, dv_0 + \int w P_0 w \, dv_0 \\
&\quad + \frac{1}{2}(3\delta - 2) Y[w].
\end{aligned}
$$

Note that F_δ is scale-invariant; i.e., $F_\delta[w + c] = F_\delta[w]$ for any constant c. By (2.7) and (2.8) the corresponding Euler equation for F_δ is

$$
\gamma_1 |\eta|^2 + Q - \frac{1}{24}(3\delta - 2) \Delta R = 0 \qquad (*)_\delta
$$

which can be rewritten as either

$$
\delta \Delta R = 8\gamma_1 |\eta|^2 - 2|E|^2 + \frac{1}{6} R^2 \qquad (*)_\delta
$$

or

$$
\sigma_2(A) = \frac{\delta}{4} \Delta R - 2\gamma_1 |\eta|^2. \qquad (*)_\delta
$$

The latter way of writing $(*)_\delta$ reveals the motivation for introducing the functional F_δ. For if $\delta = 0$, then $(*)_0$ becomes $\sigma_2(A) = -2\gamma_1 |\eta|^2$. Now recall

that $\int \sigma_2(A_0)dv_0 > 0$ implies that $\gamma_1 < 0$, so in this case we conclude that $\sigma_2(A) > 0$. This observation suggests the following strategy: to construct a conformal metric with $\sigma_2(A) > 0$, it suffices to show that F_δ admits a critical point when $\delta = 0$. This approach, however, presents some serious technical difficulties. In some sense F_δ actually degenerates as $\delta \to 0$. One can see that this is the case by writing down just the highest order terms in (2.12):

$$F_\delta[w] = \int \frac{3}{2}\delta(\Delta_0 w)^2 \quad + \quad (3\delta - 2)\,\Delta_0 w |\nabla_0 w|^2 + \frac{1}{2}(3\delta - 2)|\nabla_0 w|^4$$
$$+ \quad (\text{lower order terms}).$$

When $\delta = 0$ the leading term is absent. This behavior is reflected in the Euler equation for F_δ: when $\delta \neq 0$ then $(*)_\delta$ is fourth order in the metric, but only second order when $\delta = 0$.

Instead of studying F_0 directly we instead rely on a limiting argument. That is, we begin by showing that for any sufficiently small $\delta > 0$, $(*)_\delta$ admits a smooth solution with positive scalar curvature. Even when $\delta > 0$, though, things are hardly routine: recall that once $\delta < \frac{2}{3}$ then $\gamma_3 < 0$ while $\gamma_2 > 0$, so that the existence theory of [CY-1] does not apply. The next (and most involved) step is to obtain *a priori* estimates for solutions of $(*)_\delta$ that are independent of δ. For technical reasons that we will explain at the appropriate time, the optimal estimates we can derive give $W^{2,s}$-bounds on solutions with $s < 5$. This is sufficient to apply heat equation techniques and obtain a smooth conformal metric with $\sigma_2(A) > 0$.

To lay the groundwork for our study of $(*)_\delta$, let us begin by fixing a $\delta_0 \in (0, 1)$ and defining

$S = \{\delta \in [\delta_0, 1] | (*)_\delta \text{ admits a smooth solution with positive scalar curvature}\}.$

In Section 2 we will use the continuity method to show that $S = [\delta_0, 1]$. Since δ_0 is arbitrary, we will conclude that $(*)_\delta$ always admits a smooth solution of positive scalar curvature for any $\delta \in (0, 1]$. We end this section with a preliminary result which uses the existence theory of [CY-1] for the functional determinant in order to show that S is nonempty.

PROPOSITION 2.3.   *If* $\int \sigma_2(A_0)dv_0 > 0$ *and* $Y(g_0) > 0$, *then* $1 \in S$.

Proof. When $\delta = 1$, $\gamma_3 = \frac{1}{24}$. It follows from Corollary 2.2 that there is a smooth extremal metric $g = e^{2w}g_0$ satisfying $(*)_1$. In particular,

$$\Delta R = 8\gamma_1|\eta|^2 - 2|E|^2 + \frac{1}{6}R^2.$$

Also, $\int \sigma_2(A_0)dv_0 > 0$ implies that $\gamma_1 < 0$. Thus
$$\Delta R \leq \frac{1}{6}R^2$$
on M^4. It follows from [G-1, Lemma 1.2] that $R > 0$ on M^4. $\square$

3. The regularized equation — *a priori* estimates

In this section, we will derive some *a priori* estimates for smooth solutions of the regularized equation $(*)_\delta$.

Let F_δ denote the functional as defined in (2.12). That is, F_δ is the functional (2.7) with coefficients $\gamma_1, \gamma_2, \gamma_3$ chosen as in (2.9). The main result in this section is:

THEOREM 3.1. *Suppose* $g = e^{2w}g_0$ *is a smooth solution of* $(*)_\delta$ *with positive scalar curvature, normalized so that* $\int w\, dv_0 = 0$. *Then there exist constants* C_0, C_1, C_p *etc., all depending only on* g_0, *so that*

$$(3.1) \qquad\qquad w \geq C_0,$$

$$(3.2) \quad \int [\delta(\Delta_0 w)^2 + |\nabla_0 w|^4]\, dv_0 \leq C_1, \text{ and } \int (-\Delta_0 w)\, |\nabla_0 w|^2\, dv_0 \leq C_1.$$

Moreover, for any real number α,

$$(3.3) \qquad\qquad \int e^{\alpha w}\, dv_0 \leq C_\alpha.$$

Finally, for any positive integer p, *and for* $0 < \delta \leq \frac{1}{3}$,

$$(3.4) \qquad\qquad \int |\nabla_0 w|^4\, |w|^p\, dv_0 \leq C_p.$$

We begin the proof of Theorem 3.1 with an identity.

LEMMA 3.2. *Suppose* $g = e^{2w}g_0$ *is a solution of* $(*)_\delta$. *Then for any* $\varphi \in W^{2,2}(M^4)$,

$$(3.5) \quad \int \frac{3}{2}\delta\Delta_0 w\, \Delta_0\varphi + \frac{1}{2}(3\delta - 2)\left[\Delta_0\varphi|\nabla_0 w|^2 + 2\Delta_0 w\langle\nabla_0\varphi, \nabla_0 w\rangle_0\right.$$
$$\left. +2|\nabla_0 w|^2\langle\nabla_0\varphi, \nabla_0 w\rangle_0\right]$$
$$= \int -2U_0^\delta\varphi + 2\text{Ric}_0(\nabla_0\varphi, \nabla_0 w) + \frac{1}{2}(\delta - 2)R_0\langle\nabla_0\varphi, \nabla_0 w\rangle.$$

Remark. Although we implicitly assume in the proof that w is smooth, it follows from a standard limiting argument that (3.5) is valid if $w \in W^{2,2}(M^4)$. Indeed, we shall take (3.5) as our definition of a (weak) $W^{2,2}$-solution of $(*)_\delta$.

Proof. From a straightforward computation (cf. [CY-1, (1.8)], or [BO]), w satisfies

$$(3.6) \qquad 0 = 2U_0^\delta + P_0 w + \frac{1}{2}(3\delta - 2)\{b_1(w) + b_2(w) + b_3(w)\}$$

where

$$(3.7) \qquad b_1(w) = \Delta_0^2 w + \frac{1}{3} R_0 \Delta_0 w + \frac{1}{3} \langle \nabla_0 R_0, \nabla_0 w \rangle_0,$$

$$b_2(w) = \Delta_0 |\nabla_0 w|^2 - 2(\Delta_0 w)^2 - 2\langle \nabla_0 w, \nabla_0(\Delta_0 w) \rangle_0,$$

$$b_3(w) = -2|\nabla_0 w|^2 \Delta_0 w - 2\langle \nabla_0 w, \nabla_0 |\nabla_0 w|^2 \rangle_0.$$

Therefore,

(3.8)
$$0 = \int 2U_0^\delta \varphi + \int \varphi P_0 w + \frac{1}{2}(3\delta - 2)\left\{\int \varphi b_1(w) + \int \varphi b_2(w) + \int \varphi b_3(w)\right\}.$$

Proceeding term by term, we begin with the definition of the Paneitz operator to get

$$(3.9) \quad \int \varphi P_0 w = \int \varphi \left[\Delta_0^2 w + d^*\left(\frac{2}{3} R_0 g_0 - 2\mathrm{Ric}_0\right) dw\right]$$

$$= \int \Delta_0 \varphi \Delta_0 w + \frac{2}{3} R_0 \langle \nabla_0 \varphi, \nabla_0 w \rangle_0 - 2\,\mathrm{Ric}_0(\nabla_0 \varphi, \nabla_0 w).$$

Using the definitions in (3.7) and integrating by parts we also have

$$(3.10) \quad \int \varphi b_1(w) = \int \varphi \Delta_0^2 w + \frac{1}{3} \varphi R_0 \Delta_0 w + \frac{1}{3} \varphi \langle \nabla_0 R, \nabla_0 w \rangle_0$$

$$= \int \Delta_0 \varphi \Delta_0 w - \frac{1}{3} \varphi \langle \nabla_0 R, \nabla_0 w \rangle_0 - \frac{1}{3} R_0 \langle \nabla_0 \varphi, \nabla_0 w \rangle_0$$

$$+ \frac{1}{3} \varphi \langle \nabla_0 R_0, \nabla_0 w \rangle_0$$

$$= \int \Delta_0 \varphi \Delta_0 w - \frac{1}{3} R_0 \langle \nabla_0 \varphi, \nabla_0 w \rangle_0;$$

(3.11)
$$\int \varphi b_2(w) = \int \varphi \Delta_0 |\nabla_0 w|^2 - 2\varphi(\Delta_0 w)^2 - 2\varphi\langle \nabla_0 w, \nabla_0(\Delta_0 w) \rangle_0$$

$$= \int \Delta_0 \varphi |\nabla_0 w|^2 - 2\varphi(\Delta_0 w)^2 + 2\varphi(\Delta_0 w)^2 + 2\Delta_0 w \langle \nabla_0 \varphi, \nabla_0 w \rangle_0$$

$$= \int \Delta_0 \varphi |\nabla_0 w|^2 + 2\Delta_0 w \langle \nabla_0 \varphi, \nabla_0 w \rangle_0;$$

(3.12)
$$\int \varphi b_3(w) = \int -2\varphi|\nabla_0 w|^2 \Delta_0 w - 2\varphi\langle \nabla_0 w, \nabla_0 |\nabla_0 w|^2 \rangle_0$$

$$= \int -2\varphi|\nabla_0 w|^2 \Delta_0 w + 2\varphi\Delta_0 w|\Delta_0 w|^2 + 2|\Delta_0 w|^2 \langle \nabla_0 \varphi, \nabla_0 w \rangle_0$$

$$= \int 2|\nabla_0 w|^2 \langle \nabla_0 \varphi, \nabla_0 w \rangle_0.$$

Substituting (3.9)–(3.12) into (3.8) we arrive at (3.5). $\qquad\qquad \square$

Proof of Theorem 3.1. In the following, we let C denote various constants whose value may change but depend only on g_0.

Proof of (3.1). If R denotes the scalar curvature of g, then $R > 0$ by our assumption, and by (1.10)

$$(3.13) \qquad \Delta_0 w + |\nabla_0 w|^2 + \frac{1}{6} R e^{2w} = \frac{1}{6} R_0.$$

Hence

$$(3.14) \qquad \Delta_0 w + |\nabla_0 w|^2 \leq \frac{1}{6} R_0,$$

and in particular

$$(3.15) \qquad \Delta_0 w \leq \frac{1}{6} R_0.$$

Now by Green's formula,

$$(3.16) \qquad -w(x) + \bar{w} = \int G(x,y)\, \Delta_0 w(y)\, dv_0(y),$$

where $G(x,y)$ is the Green's function for (M, g_0), and $\bar{w} = \int w\, dv_0 = 0$. Since M is a compact manifold, we may add a constant to G and assume it is positive. Then (3.1) follows from (3.15) and (3.16).

Proof of (3.2). By integrating (3.13) over M^4, we have

$$(3.17) \qquad \int |\nabla_0 w|^2 \leq C.$$

Since $\int w = 0$, by the Poincaré inequality we conclude

$$(3.18) \qquad \int w^2 \leq C.$$

Now, taking $\varphi = w$ in (3.5) we have

$$\int \frac{3}{2} \delta (\Delta_0 w)^2 + \frac{3}{2}(3\delta - 2)\, \Delta_0 w |\nabla_0 w|^2 + (3\delta - 2)|\nabla_0 w|^4$$

$$= \int -2 U_0^\delta w + 2\mathrm{Ric}_0(\nabla_0 w, \nabla_0 w) + \frac{1}{2}(\delta - 2) R_0 |\nabla_0 w|^2.$$

Using (3.17) and (3.18) we conclude

$$(3.19) \quad \int \frac{3}{2} \delta (\Delta_0 w)^2 + \frac{3}{2}(3\delta - 2)\, \Delta_0 w |\nabla_0 w|^2 + (3\delta - 2)\,|\nabla_0 w|^4 \leq C.$$

There are now two cases to consider. First, suppose that $\delta \in \left[\frac{2}{3}, 1\right]$, i.e., that $3\delta - 2 \in [0,1]$. It then follows from the inequality

$$\frac{3}{2}(3\delta - 2)xy \geq -\frac{9}{16}(3\delta - 2)x^2 - (3\delta - 2)y^2$$

that

$$\int \frac{3}{16}\,(6-\delta)\,(\Delta_0 w)^2 \;\le\; C$$

which implies

(3.20) $$\int (\Delta_0 w)^2 \;\le\; C.$$

On the other hand, suppose $\delta \in (0, \frac{2}{3})$. Then $3\delta - 2 \in (-2, 0)$, and by (3.13)

$$|\nabla_0 w|^2\,\Delta_0 w \;+\; |\nabla_0 w|^4 \;\le\; \frac{1}{6}\,R_0 |\nabla_0 w|^2$$

whence

$$\int (3\delta - 2)\,\Delta_0 w |\nabla_0 w|^2 \;\ge\; \int -(3\delta - 2)\,|\nabla_0 w|^4 \;-\; C.$$

Substituting this into (3.19) gives

(3.21) $$\int \frac{3}{2}\,\delta(\Delta_0 w)^2 \;-\; \frac{1}{2}\,(3\delta - 2)\,|\nabla_0 w|^4 \;\le\; C,$$

and

$$\int \frac{3}{2}\,\delta(\Delta_0 w)^2 \;-\; \frac{1}{2}\,(3\delta - 2)\,\Delta_0 w |\nabla_0 w|^2 \;\le\; C.$$

Finally, using (3.14) again we observe that

$$\begin{aligned}
\int |\nabla_0 w|^4 \;&\le\; \int -\Delta_0 w |\nabla_0 w|^2 \;+\; \frac{1}{6}\,R_0 |\nabla_0 w|^2 \\
&\le\; \left(\int (\Delta_0 w)^2\right)^{\frac{1}{2}} \left(\int |\nabla_0 w|^4\right)^{\frac{1}{2}} \;+\; C,
\end{aligned}$$

which implies

(3.22) $$\int |\nabla_0 w|^4 \;\le\; \int (\Delta_0 w)^2 \;+\; C.$$

To complete the proof of (3.2), notice that when $\delta \in \left[\frac{2}{3}, 1\right]$, then (3.2) follows from (3.20) and (3.22). If $\delta \in [\frac{1}{3}, \frac{2}{3})$, then (3.2) follows from (3.22) and the first half of (3.21). If $\delta \in (0, \frac{1}{3}]$, then (3.2) follows from the first and second half of (3.21). □

Proof of (3.3). This is a direct consequence of (3.2) and the sharp Sobolev embedding theorem of Moser [M] - Trudinger [T]: if $w \,\varepsilon\, W_0^{1,n}(\Omega)$, then w is in the Orliz class $e^{L^{\frac{n}{n-1}}}(\Omega)$ for any bounded domain Ω in $\mathbb{R}^n$. In particular, $e^{\alpha w}$ is integrable for any α. Trudinger's result was later generalized ([F], [BCY]) to functions $w\varepsilon W^{1,n}(M^n)$ with $\int w\,dv_0 = 0$ on (M^n, g_0), with (M^n, g_0) any compact Riemannian manifold.

Proof of (3.4). We will prove the statement inductively on p. We first observe that by (3.1), we may replace w with $w + C_0$ and so assume that $w \ge 0$. We now substitute $\varphi = w^p$ in (3.5) and call the expression (3.5)$_p$.

Integrating by parts on the left-hand side of $(3.5)_p$, we get for $\delta < \frac{2}{3}$,

(3.23)

$$
\begin{aligned}
\text{L.H.S. of } (3.5)_p \ &= \ \frac{3\delta}{2} p \int (\Delta_0 w) \left[\Delta_0 w \, w^{p-1} + (p-1) |\nabla_0 w|^2 \, w^{p-2} \right] \\
&\quad + \frac{1}{2} (3\delta - 2) p \int \Big[3\Delta_0 w |\nabla_0 w|^2 \, w^{p-1} \\
&\quad\quad + 2 |\nabla_0 w|^4 \, w^{p-1} + (p-1) |\nabla_0 w|^4 \, w^{p-2} \Big] \\
&= \ \frac{3\delta}{2} p \int (\Delta_0 w)^2 \, w^{p-1} + \frac{3\delta}{2} p(p-1) \int (\Delta_0 w) |\nabla_0 w|^2 \, w^{p-2} \\
&\quad + \frac{1}{2}(2 - 3\delta)\, p \left[2 \int (-\Delta_0 w - |\nabla_0 w|^2) |\nabla_0 w|^2 \, w^{p-1} \right. \\
&\quad\quad \left. + \int (-\Delta_0 w) |\nabla_0 w|^2 \, w^{p-1} \right] \\
&\quad - \frac{1}{2}(p-1)(2 - 3\delta) \int |\nabla_0 w|^4 \, w^{p-2} \\
&\geq \ I_p + II_p - \frac{1}{6}(2 - 3\delta)\, p \int R_0 |\nabla_0 w|^2 \, w^{p-1}, \quad \text{(by (3.14))}
\end{aligned}
$$

where

(3.24)
$$
I_p = \frac{3\delta}{2} p \int (\Delta_0 w)^2 \, w^{p-1} + \frac{1}{2}(2 - 3\delta) p \int (-\Delta_0 w) |\nabla_0 w|^2 w^{p-1},
$$

(3.25)
$$
II_p = \frac{3\delta}{2} p(p-1) \int (\Delta_0 w) |\nabla_0 w|^2 \, w^{p-2} - \frac{1}{2}(p-1)(2 - 3\delta) \int |\nabla_0 w|^4 \, w^{p-2}.
$$

On the right-hand side of $(3.5)_p$ we have

(3.26)
$$
\begin{aligned}
\text{R.H.S. of } (3.5)_{\mathrm{p}} \ &= \ \int -2U_0^\delta \, w^p + 2p \int \mathrm{Ric}_0 \, |\nabla_0 w|^2 \, w^{p-1} \\
&\quad + \frac{1}{2}(\delta - 2) p \int R_0 |\nabla_0 w|^2 \, w^{p-1} \\
&\lesssim \ \int w^p + p \left(\int |\nabla_0 w|^4 \right)^{1/2} \left(\int w^{2(p-1)} \right)^{1/2} \\
&\lesssim \ C_p \quad \text{(by (3.2))}.
\end{aligned}
$$

Combining (3.23) and (3.26), we conclude that

(3.27)
$$
I_p + II_p \lesssim C_p.
$$

We now claim that

(3.28)
$$
-II_p \lesssim C_p \, I_{p-1} + C_p \leq C_p, \qquad \text{for } p \geq 2.
$$

To see (3.28), we first observe that from (3.14), we have

(3.29)
$$
\begin{aligned}
\int |\nabla_0 w|^4 \, w^{p-1} \ &\leq \ \int \left(\frac{1}{6} R_0 - \Delta_0 w \right) |\nabla_0 w|^2 \, w^{p-1} \\
&\leq \ \int (-\Delta_0 w) |\nabla_0 w|^2 \, w^{p-1} + C_p.
\end{aligned}
$$

Thus for $p \geq 2$, $\delta < 2/3$,

$$
\begin{aligned}
-II_p \;&\lesssim\; C_p \int (-\Delta_0 w)\, |\nabla_0 w|^2\, w^{p-2} + C_p \\
&\lesssim\; C_p\, I_{p-1} + C_p.
\end{aligned}
$$

When $p = 2$, I_1 is bounded via (3.2), thus $-II_2$ is bounded and hence I_2 is bounded via (3.27). Thus it is clear we can establish (3.28) and (3.27) inductively for all $p \geq 2$. Also, note that the constant C_p will depend on $(\frac{2}{3} - \delta)^{-1}$; thus we assume $\delta \leq \frac{1}{3}$ to eliminate this dependence. □

4. The regularized equation: Existence and regularity

In this section, we will show that for all sufficiently small $\delta > 0$, $(*)_\delta$ admits a smooth solution with positive scalar curvature. To accomplish this, we will apply the continuity method. Fix $\delta_0 \in (0, 1)$, and define

$S = \{\delta \in [\delta_0, 1] | (*)_\delta$ admits a smooth solution with positive scalar curvature$\}$.

Following the usual practice, we will show that $S = [\delta_0, 1]$ by arguing that S is both open and closed. Since we already saw that $1 \in S$ by Proposition 2.1, the desired result will follow.

PROPOSITION 4.1.   *If $\int \sigma_2(A_0) dv_0 > 0$ then S is open.*

Proof. The proof of this fact relies (as usual) on a perturbation result. Consequently, we will need to study the linearized problem.

LEMMA 4.2.   *Let $\mathcal{L}_\delta$ denote the linearization of $(*)_\delta$ at a solution g of positive scalar curvature. Then for any $\varphi \in W^{2,2}$,*

$$(4.1) \qquad \langle \varphi, \mathcal{L}_\delta \varphi \rangle_{L^2} \geq \int \frac{3}{13}\, \delta^2 (\Delta\varphi)^2 + \frac{\delta}{16}\, R |\nabla\varphi|^2.$$

In particular, $\operatorname{Ker} \mathcal{L}_\delta = \mathbb{R}$.

Remark. The kernel of $\mathcal{L}_\delta$ is due to the scale-invariance of F_δ.

Proof. By a straightforward computation (Theorem 2.1 in [CY]),

$$(4.2) \qquad \langle \varphi, \mathcal{L}_\delta \varphi \rangle_{L^2} = \int 3\delta (\Delta\varphi)^2 - 4E(\nabla\varphi, \nabla\varphi) + (1-\delta)\, R|\nabla\varphi|^2.$$

We start with the sharp inequality of [SW, p. 234]:

$$\int -4E(\nabla\varphi, \nabla\varphi) \geq \int -4\left(\frac{\sqrt{3}}{2}\right) |E|\, |\nabla\varphi|^2.$$

By the inequality $2xy \leq \varepsilon x^2 + \varepsilon^{-1} y^2$, which holds for any $\varepsilon > 0$, it follows that

$$(4.3) \qquad \int -4E(\nabla\varphi, \nabla\varphi) \geq \int -2\varepsilon \left(\frac{\sqrt{3}}{2}\right)^2 \frac{|E|^2}{R} |\nabla\varphi|^2 - 2\varepsilon^{-1} R |\nabla\varphi|^2.$$

Since g satisfies $(*)_\delta$,

$$-|E|^2 = \frac{\delta}{2} \Delta R - \frac{1}{12} R^2 - 4\gamma_1 |\eta|^2.$$

Also, $\int \sigma_2(A_0) dv_0 > 0$ implies $\gamma_1 < 0$, so that $-|E|^2 \geq \frac{\delta}{2} \Delta R - \frac{1}{12} R^2$. Substituting this into (4.3) gives

$$(4.4) \qquad \int -4E(\nabla\varphi, \nabla\varphi) \geq \int \frac{3\varepsilon\delta}{4} \frac{\Delta R}{R} |\nabla\varphi|^2 - \left(\frac{2}{\varepsilon} + \frac{\varepsilon}{8}\right) R |\nabla\varphi|^2.$$

Integrating by parts in the first term on the right-hand side of (4.4) we get

$$\int \frac{\Delta R}{R} |\nabla\varphi|^2 = \int -\nabla R \nabla(R^{-1}) |\nabla\varphi|^2 - \frac{\nabla R}{R} \nabla |\nabla\varphi|^2$$

$$= \int \frac{|\nabla R|^2}{R^2} |\nabla\varphi|^2 - 2\nabla^2\varphi \left(\nabla\varphi, \frac{\nabla R}{R}\right).$$

From the inequality $2|\nabla^2\varphi(\nabla\varphi, \frac{\nabla R}{R})| \leq |\nabla^2\varphi|^2 + \frac{|\nabla R|^2}{R^2} |\nabla\varphi|^2$, this becomes

$$\int \frac{\Delta R}{R} |\nabla\varphi|^2 \geq \int -|\nabla^2\varphi|^2.$$

By the integrated Bochner formula,

$$\int -|\nabla^2\varphi|^2 = \int -(\Delta\varphi)^2 + E(\nabla\varphi, \nabla\varphi) + \frac{1}{4} R |\nabla\varphi|^2.$$

Therefore,

$$\int \frac{\Delta R}{R} |\nabla\varphi|^2 \geq \int -(\Delta\varphi)^2 + E(\nabla\varphi, \nabla\varphi)$$

$$+ \frac{1}{4} R |\nabla\varphi|^2.$$

Substituting this into (4.4) gives

$$\int -4E(\nabla\varphi, \nabla\varphi) \geq \int -\frac{3\varepsilon\delta}{4} (\Delta\varphi)^2 + \frac{3\varepsilon\delta}{4} E(\nabla\varphi, \nabla\varphi)$$

$$+ \left(\frac{3\varepsilon\delta}{16} - \frac{2}{\varepsilon} - \frac{\varepsilon}{8}\right) R |\nabla\varphi|^2.$$

Now take $\varepsilon = \frac{4(4-\delta)}{4-3\delta}$, which yields

$$\int -4E(\nabla\varphi, \nabla\varphi) \geq \int \frac{-12\delta(4-\delta)}{(16-3\delta^2)} (\Delta\varphi)^2$$

$$+ \frac{(3\delta^3 - 44\delta^2 + 112\delta - 64)}{(4-\delta)(16-3\delta^2)} R |\nabla\varphi|^2.$$

Substituting this inequality into (4.2), we get

$$\langle \varphi, \mathcal{L}_\delta \varphi \rangle_{L^2} \geq \int \frac{3\delta^2(4 - 3\delta)}{(16 - 3\delta^2)} (\Delta \varphi)^2 + \frac{\delta(-3\delta^3 + 18\delta^2 - 40\delta + 32)}{(4 - \delta)(16 - 3\delta^2)} R|\nabla \varphi|^2.$$

Since $\delta \in [0, 1]$, we can estimate the above expressions to arrive at

$$\langle \varphi, \mathcal{L}_\delta \varphi \rangle_{L^2} \geq \int \frac{3}{13} \delta^2 (\Delta \varphi)^2 + \frac{\delta}{16} R|\nabla \varphi|^2. \qquad \square$$

Remark. Lemma 4.2 is a generalization of [G-2, Th. A], which considered the case where $\delta = \frac{2}{3}$. This corresponds to an eigenvalue estimate for the Paneitz operator. It is remarkable that, despite the coefficient δ in the leading term of (4.2), one can still show that $\mathcal{L}_\delta$ is invertible (modulo constants) for *all $\delta > 0$.*

Define the differential operator

$$G[w] = e^{4w} \left(\sigma_2(A) - \frac{\delta}{4}\Delta R + 2\gamma_1 |\eta|^2 \right).$$

If $G[w] = 0$, then $g = e^{2w}g_0$ satisfies $(*)_\delta$. From conformal invariance we see that $\int G[w]dv_0 = 0$. Thus, $G : W_0^{2,2} \to L_0^2$, where the subscript 0 denotes functions with mean value zero. If we linearize G at a solution of $(*)_\delta$, it follows from Lemma 4.2 that the linearization is invertible.

Now suppose that $\delta_1 \in S$, and that $g_1 = e^{2w_1}g_0$ is a smooth solution of $(*)_{\delta_1}$ with positive scalar curvature. It follows from [ADN, Th. 13.1] that there is a unique (up to scaling) smooth solution of $(*)_\delta$ for all δ sufficiently close to δ_1. Moreover, since the scalar curvature of g_1 is positive, by taking solutions in a small enough $C^{2,\alpha}$-neighborhood of g_1 we may conclude that the solutions of $(*)_\delta$ will also have positive scalar curvature, for δ close enough to δ_1. It follows that S is open, and the proof of Proposition 4.1 is complete. $\qquad \square$

PROPOSITION 4.3.   *S is closed.*

Proof. The proof of Proposition 4.3 consists of two parts. First, an *a priori* estimate for solutions of $(*)_\delta$ with positive scalar curvature. A consequence of this estimate will be the following: if $\{\delta_k\}$ is a sequence in S, and $\delta_k \to \bar{\delta}$, then $(*)_{\bar{\delta}}$ admits a weak $W^{2,2}(M^4)$-solution. The second part of the proof is a local estimate which, when combined with the regularity theory for extremals of the functional determinant developed in [CGY], will allow us to conclude that this weak solution of $(*)_{\bar{\delta}}$ is actually smooth with positive scalar curvature. It then follows that S is closed.

Now, let $\{\delta_k\}$ be a sequence in S, and suppose $\delta_k \to \bar{\delta}$. For each k, let $g_k = e^{2w_k}g_0$ be the corresponding solution of $(*)_{\delta_k}$, normalized so that $\int w_k = 0$. Since $\delta_k \in S$, the scalar curvature R_k of g_k is positive. Therefore, by (3.2) we have the estimate

$$\int \delta_k (\Delta_0 w_k)^2 + |\nabla_0 w_k|^4 \le C_0.$$

From this we conclude that a subsequence of $\{w_k\}$ (also denoted $\{w_k\}$) converges (i) weakly in $W^{2,2}(M^4)$, (ii) strongly in $W^{1,2}(M^4)$, (iii) almost everywhere to $w \in W^{2,2}(M^4)$. Moreover, it is clear that $g = e^{2w} g_0$ satisfies $(*)_{\bar{\delta}}$ weakly, in the sense that (3.5) holds with $\delta = \bar{\delta}$ for every $\varphi \in W^{2,2}(M^4)$.

To see that w is smooth, we need a growth estimate on the integral of $(\Delta_0 w)^2$ on a small ball of radius r. Therefore, fix $P \in M^4$ and let $\rho > 0$ be small enough so that the geodesic ball $B(\rho)$ of radius ρ (measures in the g_0 metric) centered at P admits normal coordinates $\{x^i\}$. Then in $B(\rho)$ we also have the Euclidean metric and associated Laplacian, gradient, and volume form:

$$ds^2 = \sum_{i=1}^{4} dx^i \otimes dx^i,$$

$$\bar{\Delta} = \sum_{i=1}^{4} \left(\frac{\partial}{\partial x^i} \right)^2,$$

$$\bar{\nabla} = \frac{\partial}{\partial x^i},$$

$$dx = dx^1 \wedge \ldots \wedge dx^4.$$

For $r > 0$ sufficiently small, say $r < r_0$, let $\bar{B}(r)$ denote the Euclidean ball of radius r:

$$\bar{B}(r) = \left\{ Q \in M^4 \mid \sum_{i=1}^{4} (x^i(Q))^2 < r \right\}.$$

Now fix $r \in (0, r_0)$ and define h to be the biharmonic extension of w on $\bar{B}(r)$:

$$\begin{cases} \bar{\Delta}^2 h = 0 & \text{in} \quad \bar{B}(r), \\ \frac{\partial h}{\partial n} = \frac{\partial w}{\partial n} & \text{on} \quad \partial \bar{B}(r), \\ h = w & \text{on} \quad \partial \bar{B}(r), \end{cases}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial \bar{B}(r)$ in the Euclidean metric. Define $\varphi \in W^{2,2}(M^4)$ by

$$\varphi = \begin{cases} w - h & \text{in} \quad \bar{B}(r), \\ 0 & \text{outside of} \quad \bar{B}(r). \end{cases}$$

By (3.5),

$$\int_{\bar{B}(r)} \frac{3}{2} \bar{\delta} \Delta_0 w \, \Delta_0 (w - h)$$

$$+ \frac{1}{2} (3\bar{\delta} - 2) \Big[\Delta_0 (w - h) |\nabla_0 w|^2 + 2 \Delta_0 w \langle \nabla_0 (w - h), \nabla_0 w \rangle_0$$

$$+ 2 |\nabla_0 w|^2 \langle \nabla_0 (w - h), \nabla_0 w \rangle_0 \Big]$$

$$= \int_{\bar{B}(r)} -2U_0^{\delta}(w - h) + 2\mathrm{Ric}_0(\nabla_0(w - h), \nabla_0 w)$$

$$+ \frac{1}{2}(\bar{\delta} - 2)R_0 \langle \nabla_0(w - h), \nabla_0 w \rangle_0$$

implies

(4.5) $\int_{\bar{B}(r)} \frac{3}{2}\bar{\delta}(\Delta_0 w)^2 + \frac{3}{2}(3\bar{\delta} - 2)\Delta_0 w |\nabla_0 w|^2 + (3\bar{\delta} - 2)|\nabla_0 w|^4$

$$= \int_{\bar{B}(r)} \frac{3}{2}\bar{\delta}\,\Delta_0 w \Delta_0 h + \frac{1}{2}(3\bar{\delta} - 2)\Delta_0 h |\nabla_0 w|^2$$

$$+ (3\bar{\delta} - 2)\Delta_0 w \langle \nabla_0 h, \nabla_0 w \rangle_0$$

$$+ (3\bar{\delta} - 2)|\nabla_0 w|^2 \langle \nabla_0 h, \nabla_0 w \rangle_0 - 2U_0^{\delta}(w - h)$$

$$+ 2\,\mathrm{Ric}_0(\nabla_0(w - h), \nabla_0 w) + \frac{1}{2}(\bar{\delta} - 2)R_0 \langle \nabla_0(w - h), \nabla_0 w \rangle_0.$$

Let us denote the expression on the left-hand side (respectively, right-hand side) of (4.5) by LHS (resp., RHS).

LEMMA 4.4.   (i) *There is a constant $C_1 = C_1(\bar{\delta}, g_0)$ such that*

(4.6) $\mathrm{LHS} \geq C_1\left[\int_{\bar{B}(r)} (\Delta_0 w)^2 + |\nabla_0 w|^4\right] - C_1 r^2.$

(ii) *There is a constant $C_2 = C_2(\bar{\delta}, g_0)$ such that*

(4.7) $\mathrm{RHS} \leq C_2\left[\int_{\bar{B}(r)} (\Delta_0 h)^2 + |\nabla_0 h|^4\right] + C_2 r^2.$

(iii) *There is a constant $C_3 = C_3(\bar{\delta}, g_0)$ such that*

(4.8) $\int_{\bar{B}(r)} \left[(\bar{\Delta} w)^2 + |\bar{\nabla} w|^4\right] dx \leq C_3 \int_{\bar{B}(r)} \left[(\bar{\Delta} h)^2 + |\bar{\nabla} h|^4\right] dx + C_3 r^2.$

Proof. (i) This inequality essentially follows from the arguments in the proof of (3.2) in Theorem 3.1. We begin with a claim:

Claim. $\Delta_0 w + |\nabla_0 w|^2 \leq \frac{1}{6} R_0$ almost everywhere on M^4.

To prove the claim, recall that for each k,

(4.9) $\Delta_0 w_k + |\nabla_0 w_k|^2 + \frac{1}{6} R_k e^{2w_k} = \frac{1}{6} R_0.$

Since $R_k > 0$, if we multiply both sides of (4.9) by any smooth $\psi \geq 0$ and then integrate over M^4, we obtain

$$\int \psi \Delta_0 w_k + \psi |\nabla_0 w_k| \leq \int \frac{1}{6} R_0 \psi.$$

Since $w_k \rightharpoonup w$ weakly in $W^{2,2}(M^4)$, this implies that

$$\int \psi \Delta_0 w + \psi |\Delta_0 w|^2 \leq \int \frac{1}{6} R_0 \psi$$

and the claim follows.

Now, to verify (4.6), we consider two different cases. First, suppose $\bar\delta \in \left[\frac{2}{3}, 1\right]$. Then $3\bar\delta - 2 \geq 0$, so as before we have

$$(4.10) \quad \text{LHS} = \int_{\bar B(r)} \frac{3}{2} \bar\delta (\Delta_0 w)^2 + \frac{3}{2}(3\bar\delta - 2)\Delta_0 w |\Delta_0 w|^2 + (3\bar\delta - 2)|\nabla_0 w|^4$$

$$\geq \int_{\bar B(r)} \frac{3}{2} \bar\delta (\Delta_0 w)^2 - \frac{9}{16}(3\bar\delta - 2)(\Delta_0 w)^2$$

$$- (3\bar\delta - 2)|\nabla_0 w|^4 + (3\bar\delta - 2)|\nabla_0 w|^4$$

$$= \int_{\bar B(r)} \frac{3}{16}(6 - \bar\delta)(\Delta_0 w)^2$$

$$\geq \int_{\bar B(r)} \frac{15}{16}(\Delta_0 w)^2.$$

By the claim above,

$$(4.11) \quad \int_{\bar B(r)} |\nabla_0 w|^2 \Delta_0 w + |\nabla_0 w|^4 \leq \int_{\bar B(r)} \frac{1}{6} R_0 |\nabla_0 w|^2$$

$$\leq \left(\int_{\bar B(r)} |\nabla_0 w|^4 \right)^{\frac{1}{2}} \left(\int_{\bar B(r)} \left(\frac{R_0}{6} \right)^2 \right)^{\frac{1}{2}}.$$

Notice that $w \in W^{2,2}(M^4) \subset W^{1,4}(M^4)$ implies that each integral in (4.11) is well-defined. Moreover,

$$\left(\int_{\bar B} |\nabla_0 w|^4 \right)^{\frac{1}{2}} \left(\int_{\bar B} \left(\frac{R_0}{6} \right)^2 \right)^{\frac{1}{2}} \leq C \left(\int_{\bar B(r)} \right)^{\frac{1}{2}}$$

$$\leq Cr^2,$$

and we conclude

$$(4.12) \quad \int_{\bar B(r)} |\nabla_0 w|^2 \Delta_0 w + |\nabla_0 w|^4 \leq Cr^2.$$

From (4.12) we also have

$$\int_{\bar B(r)} |\nabla_0 w|^4 \leq \int_{\bar B(r)} |\nabla_0 w|^2 (-\Delta_0 w) + Cr^2$$

$$\leq \frac{1}{2} \int_{\bar B(r)} |\nabla_0 w|^4 + \frac{1}{2} \int_{\bar B(r)} (\Delta_0 w)^2 + Cr^2$$

implies

$$(4.13) \qquad \int_{\bar{B}(r)} |\nabla_0 w|^4 \leq \int_{\bar{B}(r)} (\Delta_0 w)^2 + Cr^2.$$

Combining (4.10) and (4.13) we see that (4.6) holds when $\bar{\delta} \in \left[\frac{2}{3}, 1\right]$. If $\bar{\delta} \in \left(0, \frac{2}{3}\right)$, then $3\bar{\delta} - 2 < 0$ so from (4.12) we get

$$\int_{\bar{B}(r)} \frac{3}{2} (3\bar{\delta} - 2) \, \Delta_0 w |\nabla_0 w|^2 \geq \int_{\bar{B}(r)} -\frac{3}{2} (3\bar{\delta} - 2) |\nabla_0 w|^4 - Cr^2$$

so that

$$\text{LHS} \geq \int_{\bar{B}(r)} \frac{3}{2} \bar{\delta} (\Delta_0 w)^2 - \frac{1}{2} (3\bar{\delta} - 2) |\nabla_0 w|^4 - Cr^2.$$

This completes the proof of (i).

To prove (ii), we need to consider each term on the RHS separately. This will be considerably simplified if we begin with the following crude estimate:

$$(4.14) \qquad \text{RHS} \lesssim \int_{\bar{B}(r)} |\Delta_0 w| \, |\Delta_0 h| + \int_{\bar{B}(r)} |\nabla_0 w|^2 |\Delta_0 h|$$

$$+ \int_{\bar{B}(r)} |\Delta_0 w| \, |\nabla_0 w| \, |\nabla_0 h| + \int_{\bar{B}(r)} |\nabla_0 w|^3 \, |\nabla_0 h|$$

$$+ \int_{\bar{B}} |w - h| + \int_{\bar{B}} |\nabla_0 w| \, |\nabla_0 (w - h)|$$

where $\lesssim$ means that the inequality holds up to a multiplicative constant which depends on $\bar{\delta}$ and g_0. Using the arithmetic-geometric mean inequality we can estimate the first four integrals in (4.14) as follows:

$$(4.15) \qquad \int_{\bar{B}(r)} |\Delta_0 w| \, |\Delta_0 h| \leq \frac{1}{2} \varepsilon \int_{\bar{B}(r)} (\Delta_0 w)^2 + \frac{1}{2} \varepsilon^{-1} \int (\Delta_0 h)^2,$$

$$\int_{\bar{B}(r)} |\nabla_0 w|^2 \, |\Delta_0 h| \leq \frac{1}{2} \varepsilon \int_{\bar{B}(r)} |\nabla_0 w|^4 + \frac{1}{2} \varepsilon^{-1} \int_{\bar{B}(r)} (\Delta_0 h)^2,$$

$$\int_{\bar{B}(r)} |\Delta_0 w| \, |\nabla_0 w| \, |\nabla_0 h| \leq \frac{1}{2} \varepsilon \left[\int_{\bar{B}} (\Delta_0 w)^2 + |\nabla_0 w|^4 \right]$$
$$+ \frac{1}{8} \varepsilon^{-3} \int_{\bar{B}(r)} |\nabla_0 h|^4,$$

$$\int_{\bar{B}(r)} |\nabla_0 w|^3 \, |\nabla_0 h| \leq \varepsilon \int_{\bar{B}(r)} |\nabla_0 w|^4 + \frac{1}{8} \varepsilon^{-3} \int_{\bar{B}(r)} |\nabla_0 h|^4.$$

To estimate the last two terms in (4.14), observe that on M^4

$$\| h \|_{2,2} \le C(\| w \|_{2,2})$$

where C is independent of r (see [CGY-1, p. 237]). Therefore,

$$(4.16) \qquad \fint_{\bar{B}(r)} |w - h| \le \left(\fint_{\bar{B}(r)} (w-h)^2 \right)^{\frac{1}{2}} \left(\fint_{\bar{B}(r)} \right)^{\frac{1}{2}} \le Cr^2,$$

$$(4.17) \qquad \fint_{\bar{B}(r)} |\nabla_0 w||\nabla_0 (w-h)|$$

$$\le \left(\fint_{\bar{B}(r)} |\nabla_0 w|^4 \right)^{\frac{1}{4}} \left(\fint_{\bar{B}(r)} |\nabla_0 (w-h)|^4 \right)^{\frac{1}{4}} \left(\fint_{\bar{B}(r)} \right)^{\frac{1}{2}}$$

$$\le Cr^2.$$

By substituting (4.15)–(4.17) into (4.14) and choosing $\varepsilon > 0$ sufficiently small we get (4.7).

Finally, to prove (iii), we combine (4.6) and (4.7) to get

$$(4.18) \qquad \fint_{\bar{B}(r)} \left[(\Delta_0 w)^2 + |\nabla_0 w|^4 \right] \lesssim \fint_{\bar{B}(r)} \left[(\Delta_0 h)^2 + |\nabla_0 h|^4 \right] + r^2.$$

Then (4.8) follows from (4.18) by appealing to [CGY-1, (3.1)], which compares the Euclidean volume form, Laplacian, and gradients appearing in (4.8) with their Riemannian counterparts in (4.18). The details will be omitted. $\qquad \square$

Inequality (4.8) is precisely the conclusion of [CGY-1, Lemma 3.4]. We can therefore apply the subsequent arguments of [CGY-1] to conclude that $w \in C^\infty(M^4)$. Moreover, it follows from the claim in Lemma 4.4 that the scalar curvature R of $g = e^{2w}g_0$ is nonnegative. Since g satisfies $(*)_{\bar{\delta}}$, the scalar curvature satisfies

$$\bar{\delta}\Delta R \; = \; 8\gamma_1 |\eta|^2 - 2|E|^2 + \frac{1}{6}R^2$$

$$\le \; \frac{1}{6}R^2.$$

Thus, by the minimum principle, $R > 0$ on M^4. This completes the proof of Proposition 4.3. $\qquad \square$

Combining Propositions 4.1 and 4.3 we conclude:

COROLLARY 4.5.   *If $\int \sigma_2(A_0)dv_0 > 0$ and $Y(g_0) > 0$ then for each $\delta > 0$, $(*)_\delta$ admits a smooth solution with positive scalar curvature.*

5. *A priori* $W^{2,3}$ **estimates**

Our goal in this section is to establish the following *a priori* estimate:

THEOREM 5.1.   *Let $g = e^{2w}g_0$ be a solution of $(*)_\delta$ with positive scalar curvature, normalized so that $\int w \, dv_0 = 0$, and assume*

$$(5.1) \qquad \int \sigma_2(A_0) dv_0 \; = \; \int \sigma_2(A) dv \; > \; 0.$$

Then there are constants $C = C(g_0)$ and $0 < \delta_0 < 1$ such that

$$(5.2) \qquad \int |\nabla_0^2 w|^3 dv_0 \; + \; \int |\nabla_0 w|^{12} dv_0 \; \leq \; C$$

for $0 < \delta < \delta_0$. In particular, for any $\alpha \in (0, \frac{2}{3})$ there is a constant $C_\alpha = C(\alpha, g_0)$ such that

$$(5.3) \qquad \qquad \qquad \qquad ||w||_{C^\alpha} \leq C_\alpha.$$

The proof of (5.2) is quite involved, and will be obtained through a series of lemmas and propositions. The basic estimate we will need is:

PROPOSITION 5.2.   *Under the same hypotheses of Theorem 5.1, there is a constant $C = C(g_0)$ such that*

$$(5.4) \qquad \int \left(\frac{R}{6}\right)^3 dv \; \leq \; (1 + C\delta) \int |\nabla w|^6 dv \; + \; C \int R^2 dv \; + \; C$$

for δ sufficiently small.

Proof. We begin with some notational conventions. First, all integrals (unless otherwise specified) are with respect to the volume form dv of g. With this understood, we will suppress it from now on. Second, our calculations will sometimes be facilitated by introducing local coordinates. These coordinates are assumed to be normal at some point, and certain identities involving co-variant derivatives are understood to hold only at that point. For this reason we will not bother to distinguish between raised and lowered indices; all indices will be subscripts. For example, the gradient of a function will be denoted by $\nabla_j \varphi$, and by our conventions $\text{Ric}(\nabla\varphi, \nabla\varphi) = R_{ij}\nabla_i\varphi\nabla_j\varphi$ in local coordinates. Finally, C denotes a constant which depends at most on g_0.

Our arguments are somewhat imitative of the *a priori* C^2-estimates for Monge-Ampère equations as described in [CNS-1], [CNS-2], [Ev], and [K]. However, the estimates in these references are pointwise in nature and involve the maximum principle. Since our regularized equation is fourth order, such techniques cannot work for us. Instead, we rely on integral estimates which (as we will see) present their own difficulties. Since our calculations are quite involved, it may be helpful to begin with an overview of the argument.

We begin with a simple identity. Let $f \in C^\infty(M^4)$. Then by the divergence theorem,

$$
\begin{aligned}
0 &= \int \nabla_i(S_{ij}\,\nabla_j f) \\
&= \int \nabla_i S_{ij}\,\nabla_j f \,+\, S_{ij}\,\nabla_i\nabla_j f.
\end{aligned}
$$

Also, the contracted second Bianchi identity implies that S is divergence-free: $\nabla_i S_{ij} = \nabla_i(-R_{ij} + \frac{1}{2}\,Rg_{ij}) = -\nabla_i R_{ij} + \frac{1}{2}\nabla_j R = 0$. Therefore,

$$(5.5) \qquad 0 = \int S_{ij}\,\nabla_i\nabla_j f, \text{ for any } f \in C^\infty(M^4).$$

We will apply (5.5) to two different choices of f, resulting in two different inequalities. First, we let $f = R$. Then differentiating $(*)_\delta$ twice and using (5.5) we obtain the inequality (see (5.25))

$$(5.6) \qquad 0 = \int S_{ij}\,\nabla_i\nabla_j R \geq \int 6\,\mathrm{tr}\,E^3 + \frac{1}{12}\,R^3 + \text{(lower order terms)},$$

where $\mathrm{tr}\,E^3 = E_{ij}E_{ik}E_{jk}$. Next, we let $f = 12|\nabla w|^2$, resulting in the inequality

$$(5.7) \qquad
\begin{aligned}
0 &= \int S_{ij}\nabla_i\nabla_j(12|\nabla w|^2) \\
&\geq \int -6\mathrm{tr}\,E^3 + \frac{1}{12}\,R^3 - 6\,R|\nabla w|^4 + \text{(l.o.t.)}.
\end{aligned}
$$

Adding (5.6) to (5.7), we see that the term $\mathrm{tr}\,E^3$ cancels to give

$$\int \left(\frac{R}{6}\right)^3 \leq \int \left(\frac{R}{6}\right)|\nabla w|^4 + (\text{ l.o.t.}),$$

and (5.4) can be shown to follow from this inequality. A further argument is needed to derive (5.2), but this brief overview should provide the reader with a rough guide to the estimates which follow. To begin, let

$$(5.8) \qquad I = \int S_{ij}\,\nabla_i\nabla_j R.$$

By (5.5), $I = 0$. Also, we have:

PROPOSITION 5.3.

$$(5.9) \qquad I \geq \int 3 \left(|\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right) + 6 \operatorname{tr} E^3 + \frac{1}{12} R^3 - CR^2 - C.$$

Proof. Inequality (5.9) is a consequence of the following fundamental identity:

LEMMA 5.4.   *Let* (M^4, g) *be any Riemannian 4-manifold. Then*

$$(5.10) \quad S_{ij} \nabla_i \nabla_j R = 3\Delta \sigma_2(A) + 3 \left(|\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right)$$
$$+ 6 \operatorname{tr} E^3 + R|E|^2 - 6 W_{ijk\ell} E_{ik} E_{j\ell} - 6 E_{ij} B_{ij}$$

where B_{ij} *denotes the Bach tensor.*

Proof. By (1.18),

$$\Delta E_{ij} = \frac{1}{3} \nabla_i \nabla_j R - \frac{1}{12} \Delta R g_{ij} + 2 E_{ik} E_{jk} - \frac{1}{2} |E|^2 g_{ij}$$

$$+ \frac{1}{3} R E_{ij} - 2 W_{ikj\ell} E_{k\ell} - 2 B_{ij}.$$

Thus,

$$\frac{1}{2} \Delta |E|^2 = |\nabla E|^2 + E_{ij} \Delta E_{ij}$$

$$= |\nabla E|^2 + \frac{1}{3} E_{ij} \nabla_i \nabla_j R + 2 \operatorname{tr} E^3 + \frac{1}{3} R|E|^2$$

$$- 2 W_{ikj\ell} E_{ij} E_{k\ell} - 2 B_{ij} E_{ij},$$

$$(5.11) \qquad \Delta \sigma_2(A) = \Delta \left(-\frac{1}{2} |E|^2 + \frac{1}{24} R^2 \right)$$
$$= -|\nabla E|^2 + \frac{1}{12} |\nabla R|^2 - \frac{1}{3} E_{ij} \nabla_i \nabla_j R + \frac{1}{12} R \Delta R$$
$$- 2 \operatorname{tr} E^3 - \frac{1}{3} R|E|^2 + 2 W_{ikj\ell} E_{ij} E_{k\ell} + 2 B_{ij} E_{ik}.$$

Note that $E_{ij} = -S_{ij} + \frac{1}{4} R g_{ij}$, so that (5.11) can be rewritten

$$\Delta \sigma_2(A) = - \left(|\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right) + \frac{1}{3} S_{ij} \nabla_i \nabla_j R$$
$$- 2 \operatorname{tr} E^3 - \frac{1}{3} R|E|^2 + 2 W_{ikj\ell} E_{ij} E_{k\ell} + 2 B_{ij} E_{ij},$$

and (5.10) follows. $\square$

Continuing the proof of Proposition 5.3, we integrate (5.10) over M^4:

$$(5.12) \qquad \begin{aligned} I &= \int S_{ij} \nabla_i \nabla_j R \\ &= \int 3 \left(|\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right) + 6 \operatorname{tr} E^3 + R|E|^2 \\ &\quad - 6 \, W_{ikj\ell} \, E_{ij} \, E_{k\ell} - 6 \, B_{ij} \, E_{ij}. \end{aligned}$$

Now,

$$|W_{ikj\ell} E_{ij} E_{k\ell}| \lesssim |W| \, |E|^2.$$

By the transformation law for Weyl curvature, $|W| = e^{-2w} |W_0|_0$. From (3.1), we conclude that $|W| \leq C$. Thus

$$|W_{ikj\ell} E_{ij} E_{k\ell}| \leq C|E|^2.$$

Similarly,

$$|B_{ij} E_{ij}| \leq |E| \, |B|,$$

and $|B| = e^{-4w} |B_0|_0 \leq C$. Thus, the last two terms in (5.12) can be estimated by

$$(5.13) \qquad \begin{aligned} \int &-6 \, W_{ikj\ell} \, E_{ij} \, E_{k\ell} - 6 \, B_{ij} \, E_{ij} \\ &\gtrsim \int -|E|^2 - |E| \\ &\gtrsim - \int (|E|^2 + C). \end{aligned}$$

Note that the estimate (3.1) implies that the volume of g has uniform upper and lower bounds: $\int dv \sim 1$. It is therefore irrelevant whether we place the constant in (5.13) inside the integrand or outside.

By (5.1),

$$0 < \int \sigma_2(A) = \int -\frac{1}{2} |E|^2 + \frac{1}{24} R^2,$$

so that $\int -|E|^2 \geq -\frac{1}{12} \int R^2$. Substituting this inequality into (5.13) and combining with (5.12) we obtain

$$(5.14) \qquad \begin{aligned} I &\geq \int 3 \left(|\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right) + 6 \operatorname{tr} E^3 + R|E|^2 \\ &\quad - C R^2 - C. \end{aligned}$$

The conclusion of Proposition 5.3 will then follow from (5.14) along with part (i) of the following lemma:

LEMMA 5.5. (i) $\int R|E|^2 \geq \int \frac{1}{12} R^3 - C R^2 - C$,
(ii) $\int R|E|^2 \leq \int \frac{1}{2} \delta |\nabla R|^2 + \frac{1}{12} R^3$.

Proof. From $(*)_\delta$,

$$\int \delta R \, \Delta R \;=\; \int R \left[8\gamma_1 |\eta|^2 - 2|E|^2 + \frac{1}{6} R^2 \right]$$

$$\Longrightarrow \int R|E|^2 \;=\; \int \frac{1}{2} \delta |\nabla R|^2 + 4\gamma_1 R|\eta|^2 + \frac{1}{12} R^3.$$

Since $\gamma_1 < 0$ by (5.1), (ii) is immediate. To see (i), observe that $|\eta|^2 = e^{-4w}|\eta|_0^2 \leq C$, so that $\int 4\gamma_1 R|\eta|^2 \gtrsim \int -R^2 - 1$. $\qquad\square$

In the next two lemmas we undertake some precise estimates of the terms in (5.9).

LEMMA 5.6. *For any $p \geq 0$,*

$$(5.15) \quad \int 3 R^p \left(|\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right)$$

$$\geq \int \frac{3}{2} \delta R^{p-1} (\Delta R)^2 + \frac{3}{2} \delta p \, R^{p-2} |\nabla R|^2 \Delta R$$

$$+ 12\gamma_1 R^{p-1} \langle \nabla R, \nabla |\eta|^2 \rangle - 12\gamma_1 R^{p-2} |\eta|^2 \, |\nabla R|^2.$$

Proof. Differentiating $(*)_\delta$, we obtain

$$0 = \delta \nabla (\Delta R) - 8\gamma_1 \nabla |\eta|^2 + 4|E|\nabla |E| - \frac{1}{3} R\nabla R.$$

Now take the inner product of both sides with $R^{p-1}\nabla R$ and integrate:

$$(5.16) \quad 0 \;=\; \int \delta R^{p-1} \langle \nabla R, \nabla(\Delta R) \rangle - 8\gamma_1 R^{p-1} \langle \nabla R, \nabla |\eta|^2 \rangle$$

$$+ 4R^{p-1}|E|\langle \nabla R, \nabla |E| \rangle - \frac{1}{3} R^p |\nabla R|^2.$$

Using the arithmetic-geometric mean inequality (henceforth referred to as the AGM inequality), we obtain

$$\int 4R^{p-1}|E|\langle \nabla R, \nabla |E| \rangle \;\leq\; \int 2R^p |\nabla |E||^2 + 2R^{p-2}|E|^2|\nabla R|^2.$$

By Kato's inequality $|\nabla |E||^2 \leq |\nabla E|^2$,

$$(5.17) \quad \int 4R^{p-1}|E|\langle \nabla R, \nabla |E| \rangle \;\leq\; \int 2R^p |\nabla E|^2 + 2R^{p-2}|E|^2|\nabla R|^2.$$

We now use $(*)_\delta$ to substitute $|E|^2$ into the last term of (5.17):

$$(5.18) \quad \int 2R^{p-2}|E|^2 \, |\nabla R|^2$$

$$= \int 2R^{p-2} \left\{ -\frac{\delta}{2} \Delta R + 4\gamma_1 |\eta|^2 + \frac{1}{12} R^2 \right\} |\nabla R|^2$$

$$= \int -\delta R^{p-2} |\nabla R|^2 \, \Delta R + 8\gamma_1 R^{p-2}|\eta|^2 |\nabla R|^2 + \frac{1}{6} R^p |\nabla R|^2.$$

By (5.17) and (5.18),

$$(5.19) \quad \int 4R^{p-1}|E|\langle \nabla R, \nabla |E|\rangle \;\leq\; \int 2R^p|\nabla E|^2 + \frac{1}{6}R^p|\nabla R|^2$$
$$- \delta R^{p-2}|\nabla R|^2 \Delta R + 8\gamma_1 R^{p-2}|\eta|^2|\nabla R|^2.$$

Returning to (5.16), we integrate by parts in the first term:

$$(5.20) \quad \int \delta R^{p-1}\langle \nabla R, \nabla(\Delta R)\rangle = \int -\delta R^{p-1}(\Delta R)^2 - \delta(p-1)R^{p-2}|\nabla R|^2\,\Delta R.$$

Substituting (5.19) and (5.20) into (5.16), multiplying by $\frac{3}{2}$, then rearranging terms, we get (5.15). $\qquad \square$

COROLLARY 5.7.

$$(5.21) \quad \int 3\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) \geq \int \frac{3}{2}\delta\frac{(\Delta R)^2}{R} - C.$$

Proof. If we take $p = 0$ in (5.15), then

$$(5.22) \quad \int 3\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) \geq \int \frac{3}{2}\delta\frac{(\Delta R)^2}{R} + 12\gamma_1 R^{-1}\langle \nabla R, \nabla|\eta|^2\rangle)$$
$$- 12\gamma_1 R^{-2}|\eta|^2|\nabla R|^2.$$

Since $\gamma_1 < 0$, by the AGM inequality,

$$(5.23) \quad 12\gamma_1 R^{-1}\langle \nabla R, \nabla|\eta|^2\rangle - 12\gamma_1 R^{-2}|\eta|^2\,|\nabla R|^2$$
$$= 24\gamma_1 R^{-1}|\eta|\langle \nabla R, \nabla|\eta|\rangle - 12\gamma_1 R^{-2}|\eta|^2|\nabla R|^2$$
$$\geq 12\gamma_1 |\nabla|\eta||^2.$$

Now,

$$(5.24) \quad \int |\nabla|\eta||^2\,dv = \int e^{2w}|\nabla_0(e^{-2w}|\eta|_0)|^2\,dv_0$$
$$\lesssim \int e^{-2w}\,dv_0 + \int e^{-2w}|\nabla_0 w|^2\,dv_0$$
$$\lesssim 1 + \int |\nabla_0 w|^2\,dv_0 \leq C,$$

where the last inequality follows from (3.1). Then (5.21) follows from (5.22)–(5.24). $\qquad \square$

COROLLARY 5.8.

$$(5.25) \quad I \geq \int \frac{3}{2}\delta\frac{(\Delta R)^2}{R} + 6\mathrm{tr}\,E^3 + \frac{1}{12}R^3 - CR^2 - C.$$

Proof. This follows from (5.9) and (5.21). $\qquad \square$

The next result amounts to a technical lemma, which will be useful in our subsequent estimates.

LEMMA 5.9. *If $\delta < \frac{1}{2}$, then*

(5.26)
$$\int \delta |\nabla R|^2 \lesssim \int \delta R^3 + R^2 + 1.$$

Proof. Since E is trace-free, we have the sharp inequality

$$6\mathrm{tr}\, E^3 \geq -\frac{6}{\sqrt{3}} |E|^3$$

Thus,

$$6\mathrm{tr}\, E^3 + R|E|^2 \geq -\frac{6}{\sqrt{3}} |E|^3 + R|E|^2$$

$$= |E|^2 \left(-2\sqrt{3}|E| + R \right).$$

Using the AGM inequality

$$-2\sqrt{3}\,|E| = -2(\sqrt{6}\,|E|R^{-1/2}) \left(\frac{1}{\sqrt{2}} R^{1/2} \right) \geq -6|E|^2 R^{-1} - \frac{1}{2} R,$$

we get

(5.27)
$$\int 6\mathrm{tr}\, E^3 + R|E|^2 \geq \int \frac{|E|^2}{R} \left(-6|E|^2 + \frac{1}{2} R^2 \right)$$

$$= \int \frac{|E|^2}{R} (12\sigma_2(A))$$

$$= \int \frac{|E|^2}{R} (3\delta \Delta R - 24\gamma_1 |\eta|^2)$$

$$= \int 3\delta \frac{\Delta R}{R} |E|^2 - 24\gamma_1 \frac{|E|^2}{R} |\eta|^2$$

$$\geq \int 3\delta \frac{\Delta R}{R} |E|^2,$$

the last line following from the fact that $\gamma_1 < 0$. Using the AGM inequality once again, we obtain

$$\int 3\delta \frac{\Delta R}{R} |E|^2 \geq \int -\frac{1}{2} \delta \frac{(\Delta R)^2}{R} - \frac{9}{2} \delta \frac{|E|^4}{R},$$

and substitution of $(*)_\delta$ into the last term above gives

$$\int 3\delta \frac{\Delta R}{R} |E|^2 \geq \int -\frac{1}{2} \delta \frac{(\Delta R)^2}{R} - \frac{9}{2} \delta \frac{|E|^2}{R} \left(-\frac{\delta}{2} \Delta R + 4\gamma_1 |\eta|^2 + \frac{1}{12} R^2 \right)$$

$$= \int -\frac{1}{2} \delta \frac{(\Delta R)^2}{R} + \frac{9}{4} \delta^2 \frac{\Delta R}{R} |E|^2 - 18\delta\gamma_1 \frac{|E|^2}{R} |\eta|^2$$

$$- \frac{3}{8} \delta R |E|^2.$$

Combining terms, we have

$$(5.28) \quad \int 3\delta \left(1 - \frac{3}{4}\delta\right) \frac{\Delta R}{R} |E|^2 \geq \int -\frac{1}{2}\delta \frac{(\Delta R)^2}{R} - \frac{3}{8}\delta R|E|^2$$

$$\implies \int 3\delta \frac{\Delta R}{R} |E|^2 \geq \int -\frac{\delta}{2}\left(1 - \frac{3}{4}\delta\right)^{-1} \frac{(\Delta R)^2}{R} - \frac{3}{8}\delta(1 - \frac{3}{4}\delta)^{-1} R|E|^2.$$

Substitution of (5.28) into (5.27) gives

$$(5.29) \quad \int 6\mathrm{tr}\, E^3 + R|E|^2 \geq \int -\frac{\delta}{2}\left(1 - \frac{3}{4}\delta\right)^{-1} \frac{(\Delta R)^2}{R}$$

$$-\frac{3}{8}\delta\left(1 - \frac{3}{4}\delta\right)^{-1} R|E|^2.$$

We now substitute (5.29) into (5.25) and obtain

$$0 = I \geq \int \delta \frac{(2 - 3\delta)}{(4 - 3\delta)} \frac{(\Delta R)^2}{R}$$

$$-\frac{3}{2}\delta(4 - 3\delta)^{-1} R|E|^2 - CR^2 - C,$$

when $\delta < \frac{1}{2}$. Note that we can roughly estimate the terms above by

$$0 \geq \int \frac{\delta}{8} \frac{(\Delta R)^2}{R} - \delta R|E|^2 - CR^2 - C.$$

By Lemma 5.5 (ii) this implies

$$(5.30) \quad 0 \geq \int \frac{\delta}{8} \frac{(\Delta R)^2}{R} - \frac{\delta^2}{2}|\nabla R|^2 - \frac{\delta}{12} R^3 - CR^2 - C.$$

Finally, notice that

$$\int 2\delta\, |\nabla R|^2 = \int -2\delta R\Delta R$$

$$\leq \int \frac{\delta}{8} \frac{(\Delta R)^2}{R} + 8\delta R^3$$

$$\implies \int \frac{\delta}{8} \frac{(\Delta R)^2}{R} \geq \int 2\delta|\nabla R|^2 - 8\delta R^3.$$

Substituting this into (5.30) we conclude

$$0 \geq \int 2\delta\left(1 - \frac{\delta}{4}\right) |\nabla R|^2 - C\delta R^3 - CR^2 - C. \qquad \square$$

Now define $V = \frac{1}{2}|\nabla w|^2$. By (5.5),

$$(5.31) \quad \mathrm{II} \equiv \int S_{ij}\nabla_i\nabla_j V = 0.$$

LEMMA 5.10.

$$(5.32) \quad S_{ij} \nabla_i \nabla_j V = S_{ij} \nabla_i \nabla_k w \nabla_j \nabla_k w - \frac{1}{2} \nabla_k w \nabla_k A_{ij} S_{ij}$$

$$+ \frac{1}{2} \nabla_k w \, \nabla_k A_{ij}^0 \, S_{ij} - S_{ij} \, \nabla_i |\nabla w|^2 \, \nabla_j w$$

$$+ \frac{1}{2} R \langle \nabla w, \nabla |\nabla w|^2 \rangle + R_{ikjm} S_{ij} \, \nabla_m w \nabla_k w,$$

where R_{ikjm} denotes the components of the curvature tensor of g.

Proof. Clearly, $\nabla_j V = \nabla_j \left(\frac{1}{2} |\nabla w|^2 \right) = \nabla_j \nabla_k w \nabla_k w$. Thus, $\nabla_i \nabla_j V = \nabla_i \nabla_k w \nabla_j \nabla_k w + \nabla_i \nabla_j \nabla_k w \nabla_k w$. Since the Hessian is symmetric, $\nabla_i \nabla_j \nabla_k w = \nabla_i \nabla_k \nabla_j w$. Commuting derivatives, we find

$$\nabla_i \nabla_k \nabla_j w \;\; = \;\; \nabla_k \nabla_i \nabla_j w + R_{ikjm} \nabla_m w.$$

Therefore,

$$(5.33) \quad \nabla_i \nabla_j V = \nabla_i \nabla_k w \nabla_j \nabla_k w + \nabla_k \nabla_i \nabla_j w \nabla_k w + R_{ikjm} \nabla_m w \nabla_k w.$$

Note that by (1.16),

$$(5.34) \quad \nabla_i \nabla_j w = -\frac{1}{2} A_{ij} + \frac{1}{2} A_{ij}^0 - \nabla_i w \nabla_j w + \frac{1}{2} |\nabla w|^2 g_{ij}.$$

Hence,

$$\nabla_k \nabla_i \nabla_j w \;\; = \;\; -\frac{1}{2} \nabla_k A_{ij} + \frac{1}{2} \nabla_k A_{ij}^0 - \nabla_k \nabla_i w \nabla_j w$$

$$- \nabla_i w \nabla_k \nabla_j w + \frac{1}{2} \nabla_k |\nabla w|^2 \, g_{ij},$$

which we substitute into (5.33) to get

$$(5.35) \quad \nabla_i \nabla_j V \;\; = \;\; \nabla_i \nabla_k w \, \nabla_j \nabla_k w - \frac{1}{2} \nabla_k w \, \nabla_k A_{ij} + \frac{1}{2} \nabla_k w \, \nabla_k A_{ij}^0$$

$$- \nabla_i \nabla_k w \nabla_j w \nabla_k w - \nabla_j \nabla_k w \nabla_i w \nabla_k w$$

$$+ \frac{1}{2} \nabla_k w \nabla_k |\nabla w|^2 g_{ij} + R_{ikjm} \nabla_m w \, \nabla_k w.$$

Pairing both sides of (5.35) with S_{ij} and using the identity $S_{ij} g_{ij} = R$, we have (5.32). □

Since we will need to rewrite several of the terms in (5.32), let us denote

$$(5.36) \qquad\qquad S_{ij} \nabla_i \nabla_j V = \mathrm{II}_1 + \cdots + \mathrm{II}_6.$$

LEMMA 5.11.

$$
(5.37) \quad \text{II}_1 \equiv S_{ij} \nabla_i \nabla_k w \, \nabla_j \nabla_k w
$$

$$
= -\frac{1}{4} \operatorname{tr} E^3 + \frac{1}{48} R|E|^2 + \frac{1}{576} R^3
$$

$$
+ S_{ij} A_{jk} \nabla_i w \nabla_k w - |\nabla w|^2 \sigma_2(A) + \frac{1}{4} R|\nabla w|^4
$$

$$
- \frac{1}{2} S_{ij} A_{ik} A^0_{jk} + \frac{1}{4} S_{ij} A^0_{ik} A^0_{jk} - S_{ij} A^0_{ik} \nabla_j w \nabla_k w
$$

$$
+ \frac{1}{2} S_{ij} A^0_{ij} |\nabla w|^2 .
$$

Proof. By (5.34),

$$
(5.38) \quad \text{II}_1 = S_{ij} \nabla_i \nabla_k w \nabla_j \nabla_k w
$$

$$
= S_{ij} \left(-\frac{1}{2} A_{ik} + \frac{1}{2} A^0_{ik} - \nabla_i w \nabla_k w + \frac{1}{2} |\nabla w|^2 g_{ik} \right)
$$

$$
\cdot \left(-\frac{1}{2} A_{jk} + \frac{1}{2} A^0_{jk} - \nabla_j w \nabla_k w + \frac{1}{2} |\nabla w|^2 g_{jk} \right)
$$

$$
= \frac{1}{4} S_{ij} A_{ik} A_{jk} - \frac{1}{2} S_{ij} A_{ik} A^0_{jk} + \frac{1}{4} S_{ij} A^0_{ik} A^0_{jk}
$$

$$
+ S_{ij} A_{jk} \nabla_i w \nabla_k w - S_{ij} A^0_{ik} \nabla_j w \nabla_k w - \frac{1}{2} S_{ij} A_{ij} |\nabla w|^2
$$

$$
+ \frac{1}{2} S_{ij} A^0_{ij} |\nabla w|^2 + \frac{1}{4} R|\nabla w|^4 .
$$

Then (5.37) follows from (5.38) and the following two identities:

$$
(5.39) \qquad \frac{1}{4} S_{ij} A_{ik} A_{jk} = -\frac{1}{4} \operatorname{tr} E^3 + \frac{1}{48} R|E|^2 + \frac{1}{576} R^3 ,
$$

$$
(5.40) \qquad S_{ij} A_{ij} = 2 \sigma_2(A) .
$$

To prove (5.39) and (5.40) we simply use the fact that $A_{ij} = E_{ij} + \frac{1}{12} R g_{ij}$, $S_{ij} = -E_{ij} + \frac{1}{4} R g_{ij}$. $\qquad \square$

LEMMA 5.12.

$$
(5.41) \qquad \qquad \text{II}_2 \equiv -\frac{1}{2} \nabla_k w \, \nabla_k A_{ij} S_{ij}
$$

$$
= -\frac{1}{2} \langle \nabla w, \nabla \sigma_2(A) \rangle .
$$

Proof. We have

$$
\nabla_k A_{ij} S_{ij} = \left(\nabla_k E_{ij} + \frac{1}{12} \nabla_k R \, g_{ij} \right) \left(-E_{ij} + \frac{1}{4} R g_{ij} \right)
$$

$$
= -E_{ij} \nabla_k E_{ij} + \frac{1}{12} R \nabla_k R
$$

$$= \nabla_k \left(-\frac{1}{2} |E|^2 + \frac{1}{24} R^2 \right) = \nabla_k \, \sigma_2(A)$$

$$\Longrightarrow -\frac{1}{2} \nabla_k w \nabla_k A_{ij} \, S_{ij} = -\frac{1}{2} \langle \nabla w, \nabla \, \sigma_2(A) \rangle. \qquad \square$$

LEMMA 5.13.

$$(5.42) \quad \mathrm{II}_5 \;\equiv\; \frac{1}{2} R \langle \nabla w, \nabla |\nabla w|^2 \rangle$$

$$= -\frac{1}{2} R A_{ij} \nabla_i w \nabla_j w + \frac{1}{2} R A_{ij}^0 \nabla_i w \nabla_j w - \frac{1}{2} R |\nabla w|^4.$$

Proof. From (5.34),

$$\mathrm{II}_5 \;=\; \frac{1}{2} R \langle \nabla w, \nabla |\nabla w|^2 \rangle = R \nabla_i \nabla_j w \, \nabla_i w \nabla_j w$$

$$= R \left(-\frac{1}{2} A_{ij} + \frac{1}{2} A_{i_j}^0 - \nabla_i w \nabla_j w + \frac{1}{2} |\nabla w|^2 g_{ij} \right) \nabla_i w \nabla_j w$$

$$= -\frac{1}{2} R A_{ij} \nabla_i w \nabla_j w + \frac{1}{2} R A_{ij}^0 \nabla_i w \, \nabla_j w - \frac{1}{2} R |\nabla w|^4. \qquad \square$$

LEMMA 5.14.

$$(5.43) \qquad \mathrm{II}_6 \;\equiv\; R_{ikjm} \, S_{ij} \, \nabla_m w \nabla_k w$$

$$= W_{ikjm} \, S_{ij} \, \nabla_m w \, \nabla_k w - S_{ik} \, A_{jk} \, \nabla_i w \nabla_j w$$

$$+ \frac{1}{2} R \, A_{ij} \, \nabla_i w \, \nabla_j w + \sigma_2(A) \, |\nabla w|^2.$$

Proof. This follows directly from (1.2):

$$R_{ikjm} \, S_{ij} \, \nabla_m w \, \nabla_k w \;=\; \left(W_{ikjm} + \frac{1}{2} g_{ij} \, A_{km} - \frac{1}{2} g_{im} \, A_{jk} - \frac{1}{2} g_{jk} \, A_{im} \right.$$

$$\left. + \frac{1}{2} g_{km} \, A_{ij} \right) S_{ij} \, \nabla_m w \nabla_k w$$

$$= W_{ikjm} \, S_{ij} \, \nabla_m w \, \nabla_k w - S_{ik} \, A_{jk} \, \nabla_i w \nabla_j w$$

$$+ \frac{1}{2} R \, A_{ij} \, \nabla_i w \, \nabla_j w + \frac{1}{2} A_{ij} \, S_{ij} \, |\nabla w|^2,$$

and appealing to (5.40) for the last term we get (5.43). $\qquad \square$

Combining the results of Lemmas 5.10–5.14 we have:

COROLLARY 5.15.

$$(5.44) \quad S_{ij}\nabla_i\nabla_j V = -\frac{1}{4}\operatorname{tr} E^3 + \frac{1}{48}R|E|^2 + \frac{1}{576}R^3$$
$$-\frac{1}{2}\langle\nabla w, \nabla\sigma_2(A)\rangle - \frac{1}{4}R|\nabla w|^4$$
$$- S_{ij}\nabla_i|\nabla w|^2\,\nabla_j w + W_{ikjm}\,S_{ij}\,\nabla_m w\nabla_k w$$
$$- \frac{1}{2}S_{ij}A_{ik}\,A_{jk}^0 + \frac{1}{4}S_{ij}\,A_{ik}^0\,A_{jk}^0$$
$$- S_{ij}\,A_{ik}^0\,\nabla_i w\nabla_k w + \frac{1}{2}S_{ij}\,A_{ij}^0|\nabla w|^2$$
$$+ \frac{1}{2}\nabla_k w\,\nabla_k A_{ik}^0\,S_{ij} + \frac{1}{2}R\,A_{ij}^0\,\nabla_i w\nabla_j w.$$

PROPOSITION 5.16.

$$(5.45) \quad S_{ij}\,\nabla_i\nabla_j V \geq -\frac{1}{4}\operatorname{tr} E^3 + \frac{1}{48}R|E|^2 + \frac{1}{576}R^3$$
$$-\frac{1}{2}\langle\nabla w, \nabla\sigma_2(A)\rangle - \frac{1}{4}R|\nabla w|^4$$
$$- S_{ij}\nabla_i|\nabla w|^2\nabla_j w - C|\mathrm{Ric}|^2 - C|\mathrm{Ric}||\nabla w|^2 - C.$$

Proof. We begin with a claim:

Claim 5.17.

$$(5.46) \qquad\qquad |W| \leq C,$$
$$(5.47) \qquad\qquad |A^0| \leq C,$$
$$(5.48) \qquad\qquad |\nabla A^0| \leq C + C|\nabla w|.$$

The proof of (5.46) and (5.47) is a straightforward application of (3.1):

$$|W| = |W|_g = |W_0|_0\,e^{-2w} \leq C,$$
$$|A^0| = |A^0|_g = |A^0|_0\,e^{-2w} \leq C.$$

To prove (5.48), note that in any local coordinate system,

$$(5.49) \qquad \nabla_k A_{ij}^0 = \partial_k A_{ij}^0 - \Gamma_{ik}^m\,A_{mj}^0 - \Gamma_{jk}^m\,A_{im}^0$$

where the Γ_{ik}^m denote the Christoffel symbols relative to the metric g. By the transformation law for the Christoffel symbols under a conformal change of metric (see [Ei]),

$$(5.50) \qquad \Gamma_{jk}^m = \gamma_{jk}^m + \partial_j w\delta_{km} + \partial_k w\delta_{jm} - (g_0)^{ms}\,(g_0)_{jk}\partial_s w$$

where the γ_{jk}^m denote the Christoffel symbols relative to g_0. From (5.49) and (5.50) we conclude (5.48).

Using (5.46)–(5.48) we can estimate the last seven terms of (5.44) as follows:

$$
\begin{aligned}
W_{jkim}\, S_{ij} \nabla_m w \nabla_k w &\geq\ -C\,|\mathrm{Ric}|\,|\nabla w|^2, \\
-\frac{1}{2}\, S_{ij}\, A_{ik}\, A^0_{jk} &\geq\ -C\,|\mathrm{Ric}|^2, \\
\frac{1}{4}\, S_{ij}\, A^0_{ik}\, A^0_{jk} &\geq\ -C\,|\mathrm{Ric}|, \\
-S_{ij}\, A^0_{ik}\, \nabla_j w\, \nabla_k w &\geq\ -C\,|\mathrm{Ric}|\,|\nabla w|^2, \\
\frac{1}{2}\, S_{ij}\, A^0_{ij}\, |\nabla w|^2 &\geq\ -C\,|\mathrm{Ric}|\,|\nabla w|^2, \\
\frac{1}{2}\, \nabla_k w\, \nabla_k A^0_{ij}\, S_{ij} &\geq\ -C\,|\mathrm{Ric}|\,|\nabla w| - C|\mathrm{Ric}|\,|\nabla w|^2, \\
\frac{1}{2}\, R\, A^0_{ij}\, \nabla_i w \nabla_j w &\geq\ -C\,|\mathrm{Ric}|\,|\nabla w|^2,
\end{aligned}
$$

and obtain (5.45). □

PROPOSITION 5.18. *For all* $\delta > 0$ *sufficiently small,*

$$
(5.51) \qquad 0 = \mathrm{II}\ \geq\ \int -\frac{1}{4}\,\mathrm{tr}\,E^3 + \frac{1}{288}\,R^3 - \frac{1}{4}\,R|\nabla w|^4 \\
- C\,\delta R^3 - C\delta|\nabla w|^6 - C\,R^2 - C.
$$

Proof. Integrating by parts and using (1.14) we have

$$
\begin{aligned}
(5.52)\quad \int -\frac{1}{2}\langle \nabla w, \nabla \sigma_2(A)\rangle &= \int \frac{1}{2}\,\Delta w\,\sigma_2(A) \\
&= \int \frac{1}{2}\left(-\frac{R}{6} + \frac{1}{6}\,R_0\,e^{-2w} + |\nabla w|^2\right)\sigma_2(A) \\
&= \int -\frac{1}{12}\,R\,\sigma_2(A) + \frac{1}{12}\,R_0 e^{-2w}\,\sigma_2(A) \\
&\quad + \frac{1}{2}|\nabla w|^2 \sigma_2(A).
\end{aligned}
$$

Similarly, by (5.34) and (5.40),

$$
\begin{aligned}
(5.53)\quad \int -S_{ij}\,\nabla_i|\nabla w|^2 \nabla_j w &= \int |\nabla w|^2\,\nabla_i S_{ij}\,\nabla_j w + |\nabla w|^2\,S_{ij}\,\nabla_i\nabla_j w \\
&= \int |\nabla w|^2\,S_{ij}\left\{-\frac{1}{2}\,A_{ij} + \frac{1}{2}A^0_{ij} - \nabla_i w \nabla_j w \right. \\
&\qquad \left. + \frac{1}{2}|\nabla w|^2\,g_{ij}\right\} \\
&= \int -\frac{1}{2}|\nabla w|^2\,S_{ij}A_{ij} + \frac{1}{2}S_{ij}\,A^0_{ij}\,|\nabla w|^2 \\
&\qquad - |\nabla w|^2\,S_{ij}\,\nabla_i w\,\nabla_j w + \frac{1}{2}R|\nabla w|^4
\end{aligned}
$$

$$= \int -|\nabla w|^2 \sigma_2(A) + \frac{1}{2} S_{ij} A^0_{ij} |\nabla w|^2$$
$$+ |\nabla w|^2 R_{ij} \nabla_i w \nabla_j w.$$

Substituting (5.52) and (5.53) into (5.45) we get

$$(5.54) \qquad 0 = \mathrm{II} \ \geq \ \int -\frac{1}{4} \mathrm{tr} E^3 + \frac{1}{48} R|E|^2 + \frac{1}{576} R^3$$
$$-\frac{1}{12} R\sigma_2(A) + \frac{1}{12} R_0 e^{-2w} \sigma_2(A) - \frac{1}{2}|\nabla w|^2 \sigma_2(A)$$
$$+ |\nabla w|^2 R_{ij} \nabla_i w \nabla_j w - \frac{1}{4} R|\nabla w|^4 + \frac{1}{2} S_{ij} A^0_{ij} |\nabla w|^2$$
$$- C|\mathrm{Ric}|^2 - C|\mathrm{Ric}| |\nabla w|^2 - C.$$

By $(*)_\delta$,

$$\int -\frac{1}{12} R \sigma_2(A) \ = \ \int -\frac{1}{12} R \left(\frac{\delta}{4} \Delta R - 2\gamma_1 |\eta|^2 \right)$$
$$= \ \int -\frac{\delta}{48} R\Delta R + \frac{\gamma_1}{6} R|\eta|^2$$
$$= \ \int \frac{\delta}{48} |\nabla R|^2 + \frac{\gamma_1}{6} R|\eta|^2$$
$$\geq \ \int \frac{\delta}{48} |\nabla R|^2 - C|\mathrm{Ric}|^2 - C.$$

Similarly,

$$\int \frac{1}{12} R_0 e^{-2w} \sigma_2(A) \ = \ \int \frac{1}{12} R_0 e^{-2w} \left(\frac{\delta}{4} \Delta R - 2\gamma_1 |\eta|^2 \right)$$
$$= \ \int \frac{\delta}{48} R_0 e^{-2w} \Delta R - \frac{\gamma_1}{6} R_0 e^{-2w} |\eta|^2$$
$$= \ \int -\frac{\delta}{48} \langle \nabla R_0, \nabla R \rangle e^{-2w} - \frac{\delta}{48} \langle \nabla(e^{-2w}), \nabla R \rangle R_0$$
$$- \frac{\gamma_1}{6} R_0 e^{-2w} |\eta|^2$$
$$\geq \ \int -C\delta|\nabla R| - C\delta|\nabla w| |\nabla R| - C$$
$$\geq \ \int -\frac{\delta}{96} |\nabla R|^2 - C|\nabla w|^2 - C$$
$$\geq \ \int -\frac{\delta}{96} |\nabla R|^2 - C,$$

which when combined with the estimate above gives

$$(5.55) \qquad \int -\frac{1}{12} R\sigma_2(A) + \frac{1}{12} R_0 e^{-2w} \sigma_2(A) \geq \int \frac{\delta}{96} |\nabla R|^2 - C|\mathrm{Ric}|^2 - C.$$

Also,

$$
\begin{aligned}
(5.56) \quad \int -\frac{1}{2}\,|\nabla w|^2\, \sigma_2(A) \;&=\; \int -\frac{1}{2}\,|\nabla w|^2\left(\frac{\delta}{4}\,\Delta R - 2\gamma_1|\eta|^2\right)\\
&\geq\; \int -\frac{\delta}{8}\,|\nabla w|^2\,\Delta R - C|\nabla w|^2\\
&\geq\; \int \frac{\delta}{8}\,\langle\nabla|\nabla w|^2, \nabla R\rangle - C\\
&\geq\; \int -\frac{\delta}{4}\,|\nabla^2 w|\,|\nabla w|\,|\nabla R - C\\
&\geq\; \int -\frac{\delta}{96}\,|\nabla R|^2 - \frac{3}{2}\,\delta|\nabla^2 w|^2\,|\nabla w|^2 - C.
\end{aligned}
$$

LEMMA 5.19.

$$
(5.57) \qquad \int -\delta\,|\nabla^2 w|^2\,|\nabla w|^2 \;\gtrsim\; \int -\delta R^3 - \delta|\nabla w|^6 - R^2 - 1.
$$

Proof. From (5.34) we have

$$
\begin{aligned}
|\nabla^2 w|^2 \;&=\; \left| -\frac{1}{2}\,A_{ij} + \frac{1}{2}\,A^0_{ij} - \nabla_i w\nabla_j w + \frac{1}{2}\,|\nabla w|^2\,g_{ij}\right|^2\\
&\lesssim\; |A|^2 + |\nabla w|^4 + 1.
\end{aligned}
$$

Thus,

$$
\int -\delta\,|\nabla^2 w|^2\,|\nabla w|^2 \;\geq\; \int -C\delta\,|A|^2\,|\nabla w|^2 - C\delta\,|\nabla w|^6 - C\delta.
$$

Since $\sigma_2(A) = -\frac{1}{2}\,|A|^2 + \frac{1}{18}\,R^2$,

$$
\begin{aligned}
\int -\delta\,|\nabla^2 w|^2\,|\nabla w|^2 &\\
&\geq\; \int -C\delta\,|\nabla w|^2\left(-2\,\sigma_2(A) + \frac{1}{9}\,R^2\right) - C\delta\,|\nabla w|^6 - C\delta\\
&=\; \int -C\delta\,|\nabla w|^2\,\sigma_2(A) - C\delta R^2|\nabla w|^2\\
&\qquad - C\delta|\nabla w|^6 - C\delta\\
&=\; \int -C\delta\,|\nabla w|^2\left(\frac{\delta}{4}\,\Delta R - 2\gamma_1|\eta|^2\right) - C\delta R^2|\nabla w|^2\\
&\qquad - C\delta\,|\nabla w|^6 - C\delta\\
&=\; \int -C\delta^2\,|\nabla w|^2\,\Delta R - C\delta R^2\,|\nabla w|^2 - C\delta\,|\eta|^2\,|\nabla w|^2\\
&\qquad - C\delta\,|\nabla w|^6 - C\delta\\
&\geq\; \int C\delta^2\langle\nabla|\nabla w|^2, \nabla R\rangle - C\delta R^2\,|\nabla w|^2 - C\delta\,|\nabla w|^6 - C\delta\\
&\geq\; \int -C\delta^2\,|\nabla^2 w|\,|\nabla w|\,|\nabla R| - C\delta R^2\,|\nabla w|^2 - C\delta|\nabla w|^6 - C\delta
\end{aligned}
$$

$$\geq \int -C\delta^2 \, |\nabla R|^2 - C\delta^2 \, |\nabla^2 w|^2 \, |\nabla w|^2 - C\delta R^2 \, |\nabla w|^2$$
$$- C\delta \, |\nabla w|^6 - C\delta.$$

Therefore, when δ is sufficiently small,

$$\int - \delta \, |\nabla^2 w|^2 \, |\nabla w|^2 \geq \int -C\delta^2 \, |\nabla R|^2 - C\delta R^2 \, |\nabla w|^2 - C\delta \, |\nabla w|^6 - C\delta.$$

By Lemma 5.9,

$$\int - C\delta^2 \, |\nabla R|^2 \geq \int -C\delta^2 R^3 - C\delta R^2 - C\delta,$$

and since $R^2 \, |\nabla w|^2 \lesssim R^3 + |\nabla w|^6$ we conclude

$$\int - \delta \, |\nabla^2 w|^2 \, |\nabla w|^2 \gtrsim \int - \delta R^3 - \delta R^2 - \delta |\nabla w|^6 - \delta. \qquad \square$$

Substituting (5.57) into (5.56), then (5.56) and (5.55) into (5.54) we get

$$(5.58) \qquad 0 = \text{II} \;\geq\; \int -\frac{1}{4} \operatorname{tr} E^3 + \frac{1}{48} R|E|^2 + \frac{1}{576} R^3$$
$$+ |\nabla w|^2 R_{ij} \nabla_i w \nabla_j w - \frac{1}{4} R|\nabla w|^4$$
$$- C\delta R^3 - C\delta \, |\nabla w|^6 - C$$
$$+ \frac{1}{2} |\nabla w|^2 S_{ij} A_{ij}^0 - C|\text{Ric}|^2 - C|\text{Ric}| \, |\nabla w|^2.$$

We estimate the last three terms in (5.58) as follows:

$$\int \frac{1}{2} |\nabla w|^2 S_{ij} A_{ij}^0 \;\geq\; \int - C \, |\text{Ric}| \, |\nabla w|^2,$$
$$\int - C \, |\text{Ric}|^2 \;=\; \int C \left(2\sigma_2(A) - \frac{1}{3} R^2 \right)$$
$$\geq\; \int C - CR^2,$$
$$\int - |\text{Ric}| \, |\nabla w|^2 \;\geq\; \int - |\text{Ric}|^2 - |\nabla w|^4$$
$$\geq\; \int - CR^2 - C,$$

the last line following from (3.2). Therefore,

$$(5.59) \qquad 0 = \text{II} \geq \int \;-\; \frac{1}{4} \operatorname{tr} E^3 + \frac{1}{48} R|E|^2 + \frac{1}{576} R^3$$
$$+\; |\nabla w|^2 R_{ij} \, \nabla_i w \nabla_j w - \frac{1}{4} R|\nabla w|^4$$
$$-\; C\delta R^3 - C\delta|\nabla w|^6 - CR^2 - C.$$

Using (1.7) and integrating by parts, we can estimate the term involving the Ricci curvature in (5.59) as follows:

$$
\begin{aligned}
\int |\nabla w|^2 \, R_{ij} \, \nabla_i w \nabla_j w \;&\geq\; \int \frac{3}{R} \, \sigma_2(A) \, |\nabla w|^4 \\
&=\; \int \frac{3}{R} \left(\frac{\delta}{4} \Delta R - 2\gamma_1 |\eta|^2 \right) |\nabla w|^4 \\
&\geq\; \int \frac{3}{4} \delta \, \frac{\Delta R}{R} \, |\nabla w|^4 \\
&=\; \int -\frac{3}{4} \delta \, \nabla R \, \nabla(R^{-1}) \, |\nabla w|^4 - \frac{3}{4} \delta \, \frac{\nabla R}{R} \, \nabla |\nabla w|^4 \\
&=\; \int \frac{3}{4} \delta \, \frac{|\nabla R|^2}{R^2} \, |\nabla w|^4 - 3\delta \, |\nabla w|^2 \, \nabla^2 w \left(\frac{\nabla R}{R}, \nabla w \right) \\
&\geq\; \int \frac{3}{4} \delta \frac{|\nabla R|^2}{R^2} \, |\nabla w|^4 - \frac{3}{4} \delta \, \frac{|\nabla R|^2}{R^2} \, |\nabla w|^4 \\
&\quad - 3\delta |\nabla^2 w|^2 \, |\nabla w|^2 \\
&=\; \int -3\delta \, |\nabla^2 w|^2 \, |\nabla w|^2.
\end{aligned}
$$

Therefore, by Lemma 5.19,

(5.60) $$\int |\nabla w|^2 \, R_{ij} \, \nabla_i w \nabla_j w \gtrsim \int -\delta R^3 - \delta |\nabla w|^6 - R^2 - 1.$$

Substituting (5.60) into (5.59) and using Lemma 5.5(i), we arrive at (5.51). □

Through (5.51) we can now complete the proof of Proposition 5.2. We begin by adding the results of Proposition 5.18 and Corollary 5.7:

$$
\begin{aligned}
\text{(5.61)} \qquad 0 \geq \mathrm{I} + 24\,\mathrm{II} \;&=\; \int S_{ij} \, \nabla_i \nabla_j R + 24 \, S_{ij} \nabla_i \nabla_j V \\
&\geq\; \int \delta \, \frac{(\Delta R)^2}{R} + 6\mathrm{tr}\, E^3 + \frac{1}{12} R^3 \\
&\quad - 6\mathrm{tr}\, E^3 + \frac{1}{12} R^3 - 6R |\nabla w|^4 \\
&\quad - C\delta R^3 - C\delta |\nabla w|^6 - CR^2 - C \\
&=\; \int \delta \, \frac{(\Delta R)^2}{R} + \frac{1}{6} R^3 - 6R |\nabla w|^4 \\
&\quad - C\delta R^3 - C\delta |\nabla w|^6 - CR^2 - C.
\end{aligned}
$$

Dividing through by 36 and using Hölder's inequality we find

$$
\int \left(\frac{R}{6}\right)^3 \;\leq\; \int \frac{R}{6}|\nabla w|^4 + C\delta R^3 + C\delta|\nabla w|^6 + CR^2 + C
$$

$$
\leq\; \left(\int \left(\frac{R}{6}\right)^3\right)^{\frac{1}{3}} \left(\int |\nabla w|^6\right)^{\frac{2}{3}} + \int C\delta R^3 + C\delta|\nabla w|^6
$$

$$
+\, CR^2 + C.
$$

Since $xy \leq \frac{x^3}{3} + \frac{2}{3}y^{3/2}$ for $x, y \geq 0$,

$$
\int \left(\frac{R}{6}\right)^3 \leq \frac{1}{3}\int \left(\frac{R}{6}\right)^3 + \frac{2}{3}\int |\nabla w|^6 + C\int \left[\delta R^3 + \delta|\nabla w|^6 + R^2 + 1\right]
$$

which implies (5.4).

It remains to show how (5.4) leads to the $W^{2,3}$-estimate in Theorem 5.1. This requires several intermediate estimates, beginning with

PROPOSITION 5.20.

$$
(5.62) \qquad \int |\nabla^2 w|^2 |\nabla w|^2 \lesssim \int \delta|\nabla w|^6 + R^2 + 1.
$$

Proof. We will need some preparatory inequalities

LEMMA 5.21.

$$
(5.63)\quad \text{(i)}\quad \int \left(\frac{R}{6}\right)^2 |\nabla w|^2 \leq (1 + C\delta)\int |\nabla w|^6 + C\int R^2 + C,
$$

$$
(5.64)\quad \text{(ii)}\quad \int (\Delta w)^2 |\nabla w|^2 \leq \int 2\Delta w|\nabla w|^4
$$
$$
+\, C\delta R^3 + C\delta|\nabla w|^6 + CR^2 + C,
$$

$$
(5.65)\quad \text{(iii)}\quad \int |\nabla w|^6 \leq \int \frac{1}{6}R|\nabla w|^4 + C\delta R^3 + C\delta|\nabla w|^6 + CR^2 + C,
$$

$$
(5.66)\quad \text{(iv)}\quad \int \Delta w|\nabla w|^4 \lesssim \int \delta R^3 + \delta|\nabla w|^6 + R^2 + 1.
$$

Proof. (i) Using $xy \leq \frac{2}{3}x^{3/2} + \frac{1}{3}y^3$ along with (5.4), we have

$$
\int \left(\frac{R}{6}\right)^2 |\nabla w|^2 \;\leq\; \frac{2}{3}\int \left(\frac{R}{6}\right)^3 + \frac{1}{3}\int |\nabla w|^6
$$

$$
\leq\; (1 + C\delta)\int |\nabla w|^6 + \int CR^2 + C.
$$

(ii) By (1.14),

$$\int \left(\frac{R}{6}\right)^2 |\nabla w|^2 - \int |\nabla w|^6$$

$$= \int \left(-\Delta w + |\nabla w|^2 + \frac{1}{6} R_0 e^{-2w}\right)^2 |\nabla w|^2 - \int |\nabla w|^6$$

$$= \int (\Delta w)^2 |\nabla w|^2 + |\nabla w|^6 + \frac{1}{36} R_0^2 e^{-4w} |\nabla w|^2$$

$$-2\Delta w |\nabla w|^4 - \frac{1}{3} \Delta w R_0 e^{-2w} |\nabla w|^2 + \frac{1}{3} R_0 e^{-2w} |\nabla w|^4 - |\nabla w|^6$$

leads to

$$(5.67) \quad \int (\Delta w)^2 |\nabla w|^2 = \int \left[\left(\frac{R}{6}\right)^2 |\nabla w|^2 - |\nabla w|^6\right]$$

$$+ \int 2\Delta w |\nabla w|^4 - \frac{1}{36} R_0^2 e^{-4w} |\nabla w|^2$$

$$+ \frac{1}{3} \Delta w R_0 e^{-2w} |\nabla w|^2 - \frac{1}{3} R_0 e^{-2w} |\nabla w|^4.$$

The last two terms in (5.67) can be estimated using (1.14) and (3.2):

$$\int \frac{1}{3} \Delta w R_0 e^{-2w} |\nabla w|^2 \lesssim \int |\nabla w|^4 + (\Delta w)^2$$

$$\lesssim \int 1 + R^2,$$

$$\int -\frac{1}{3} R_0 e^{-2w} |\nabla w|^4 \lesssim \int |\nabla w|^4 \leq C.$$

Finally, appealing to (5.63) we get (5.64).

(iii) By (1.16) and (1.14)

$$(5.68) \quad \int 2|\nabla w|^2 R_{ij} \nabla_i w \nabla_j w = \int 2|\nabla w|^2 A_{ij} \nabla_i w \nabla_j w + \frac{1}{3} R |\nabla w|^4$$

$$= \int 2|\nabla w|^2 \left\{A_{ij}^0 - 2\nabla_i \nabla_j w - 2\nabla_i w \nabla_j w + |\nabla w|^2 g_{ij}\right\} \nabla_i w \nabla_j w$$

$$+ \frac{1}{3} R |\nabla w|^4$$

$$= \int 2|\nabla w|^2 A_{ij}^0 \nabla_i w \nabla_j w - 4|\nabla w|^2 \nabla_i \nabla_j w \nabla_i w \nabla_j w$$

$$- 2|\nabla w|^6 + \frac{1}{3} R |\nabla w|^4$$

$$= \int 2|\nabla w|^2 A_{ij}^0 \nabla_i w \nabla_j w - \nabla_i |\nabla w|^4 \nabla_i w - 2|\nabla w|^6$$

$$+ \frac{1}{3} R |\nabla w|^4$$

$$= \int 2|\nabla w|^2 A_{ij}^0 \nabla_i w \nabla_j w + \Delta w |\nabla w|^4 - 2|\nabla w|^6 + \frac{1}{3} R |\nabla w|^4$$

$$
\begin{aligned}
&= \int 2|\nabla w|^2 \, A_{ij}^0 \, \nabla_i w \nabla_j w + \left(|\nabla w|^2 - \frac{1}{6} R + \frac{1}{6} R_0 e^{-2w} \right) |\nabla w|^4 \\
&\quad - 2|\nabla w|^6 + \frac{1}{3} R |\nabla w|^4 \\
&= \int 2|\nabla w|^2 \, A_{ij}^0 \nabla_i w \nabla_j w + \frac{1}{6} R_0 e^{-2w} |\nabla w|^4 \\
&\quad - |\nabla w|^6 + \frac{1}{6} R |\nabla w|^4 \\
&\leq \int C - |\nabla w|^6 + \frac{1}{6} R |\nabla w|^4.
\end{aligned}
$$

Combining (5.68) with (5.60) we get (5.65).

(iv) Substituting (1.14) into (5.65), we obtain

$$
\begin{aligned}
\int |\nabla w|^6 &\leq \int \frac{1}{6} R |\nabla w|^4 + C\delta R^3 + C\delta |\nabla w|^6 \\
&\quad + CR^2 + C \\
&= \int \left(\frac{1}{6} R_0 e^{-2w} - \Delta w + |\nabla w|^2 \right) |\nabla w|^4 \\
&\quad + C\delta R^3 + C\delta |\nabla w|^6 + CR^2 + C \\
&\leq \int -\Delta w |\nabla w|^4 + |\nabla w|^6 + C\delta R^3 + C\delta |\nabla w|^6 \\
&\quad + CR^2 + C \\
\Longrightarrow \int \Delta w |\nabla w|^4 &\leq \int C\delta R^3 + C\delta |\nabla w|^6 + CR^2 + C. \qquad \square
\end{aligned}
$$

Now, combining (5.64) and (5.66) we find

$$
(5.69) \qquad \int (\Delta w)^2 \, |\nabla w|^2 \lesssim \int \delta R^3 + \delta |\nabla w|^6 + R^2 + 1.
$$

By the Bochner formula,

$$
\frac{1}{2} \Delta |\nabla w|^2 = |\nabla^2 w|^2 + R_{ij} \nabla_i w \nabla_j w + \langle \nabla w, \nabla(\Delta w) \rangle.
$$

Multiplying both sides by $|\nabla w|^2$ and integrating by parts give

$$
\begin{aligned}
(5.70) \quad \int |\nabla^2 w|^2 \, |\nabla w|^2 &= \int \frac{1}{2} |\nabla w|^2 \, \Delta |\nabla w|^2 - |\nabla w|^2 \, R_{ij} \nabla_i w \nabla_j w \\
&\quad - |\nabla w|^2 \, \langle \nabla w, \nabla(\Delta w) \rangle \\
&= \int -\frac{1}{2} |\nabla |\nabla w|^2|^2 - |\nabla w|^2 \, R_{ij} \nabla_i w \nabla_j w \\
&\quad + |\nabla w|^2 (\Delta w)^2 + \Delta w \langle \nabla w, \nabla |\nabla w|^2 \rangle \\
&\leq \int -\frac{1}{2} |\nabla |\nabla w|^2|^2 - |\nabla w|^2 \, R_{ij} \nabla_i w \nabla_j w \\
&\quad + |\nabla w|^2 (\Delta w)^2 + \frac{1}{2} |\nabla w|^2 (\Delta w)^2 + \frac{1}{2} |\nabla |\nabla w|^2|^2
\end{aligned}
$$

$$= \int \frac{3}{2} |\nabla w|^2 (\Delta w)^2 - |\nabla w|^2 R_{ij} \nabla_i w \nabla_j w.$$

Substituting (5.60) and (5.69) into (5.70) and appealing to (5.4) give (5.62), which concludes the proof of Proposition 5.20. □

From Lemma 5.21 we can deduce that the inequality in (5.4) can be reversed:

COROLLARY 5.21.

$$(5.71) \qquad \int |\nabla w|^6 \leq (1 + C\delta) \int \left(\frac{R}{6} \right)^3 + \int CR^2 + C.$$

Proof. By (5.65),

$$\int |\nabla w|^6 \leq \int \frac{R}{6} |\nabla w|^4 + C\delta R^3 + C\delta |\nabla w|^6 + CR^2 + C.$$

Once again using the inequality $xy \leq \frac{2}{3} x^{3/2} + \frac{1}{3} y^3$, we have

$$\int |\nabla w|^6 \leq \int \frac{1}{3} \left(\frac{R}{6} \right)^3 + \frac{2}{3} |\nabla w|^6 + C\delta R^3 + C\delta |\nabla w|^6 + CR^2 + 1,$$

which implies (5.71). □

The following reverse-Hölder inequality will be the penultimate estimate in the proof of Theorem 5.1.

PROPOSITION 5.22.

$$(5.72) \qquad \left(\int |\nabla w|^{12} \right)^{\frac{1}{4}} \lesssim \int |\nabla w|^6 + 1.$$

Proof. In the following, some of our calculations are done in the background metric g_0. For this reason, we will carefully distinguish between quantities that are given with respect to g versus g_0.

To begin, recall that the Sobolev embedding theorem implies that $W^{1,3} \hookrightarrow L^{12}$. Thus, for any $f \in W^{1,3}$,

$$\left(\int |f|^{12} \, dv_0 \right)^{1/4} \lesssim \int |\nabla_0 f|^3 \, dv_0 + \int |f|^3 \, dv_0.$$

If we take $f = |\nabla_0 w| e^{-\frac{2}{3} w}$, then

$$\int f^{12} \, dv_0 = \int |\nabla_0 w|^{12} e^{-8w} \, dv_0$$

$$= \int |\nabla w|^{12} \, dv.$$

Therefore,

(5.73)
$$\left(\int |\nabla w|^{12}\, dv\right)^{\frac{1}{4}} \lesssim \int |\nabla_0 \left(|\nabla_0 w|e^{-\frac{2}{3}w}\right)|^3 \, dv_0$$
$$+ \int |\nabla_0 w|^3\, e^{-2w}\, dv_0$$
$$= \int |e^{-\frac{2}{3}w}\nabla_0|\nabla_0 w| + |\nabla_0 w|\nabla_0\left(e^{-\frac{2}{3}w}\right)|^3\, dv_0$$
$$+ \int |\nabla_0 w|^3 e^{-2w}\, dv_0$$
$$\lesssim \int \left[|\nabla_0^2 w|^3\, e^{-2w} + |\nabla_0 w|^6\, e^{-2w} + |\nabla_0 w|^3\, e^{-2w}\right]\, dv_0$$
$$\lesssim \int \left[|\nabla_0^2 w|^3\, e^{-2w} + |\nabla_0 w|^6\, e^{-2w}\right]\, dv_0 + C.$$

Comparing the Hessian ∇^2 relative to the metric g with the Hessian ∇_0^2 relative to g_0 we have

$$|\nabla_0^2 w|^2 \lesssim e^{4w}\, |\nabla^2 w|^2 + e^{4w}|\nabla w|^4.$$

Thus,

(5.74)
$$|\nabla_0^2 w|^3\, e^{-2w} \lesssim e^{4w}|\nabla^2 w|^3 + e^{4w}|\nabla w|^6.$$

Substituting (5.74) into (5.73) we get

(5.75)
$$\left(\int |\nabla w|^{12}dv\right)^{1/4} \lesssim \int |\nabla^2 w|^3\, dv + \int |\nabla w|^6\, dv + 1.$$

By (5.34),

(5.76)
$$|\nabla^2 w|^3 \lesssim |A|^3 + |\nabla w|^6 + 1,$$

so that

(5.77)
$$\left(\int |\nabla w|^{12}\, dv\right)^{1/4} \lesssim \int \left(|A|^3 + |\nabla w|^6\right)\, dv + 1.$$

LEMMA 5.23.

(5.78)
$$\int |A|^3\, dv \lesssim \int R^3\, dv + 1.$$

Proof. First, notice $|A|^2 = |E|^2 + \frac{1}{36}R^2$ implies that

(5.79)
$$\int |A|^3\, dv \lesssim \int \left(|E|^3 + R^3\right)\, dv.$$

By $(*)_\delta$,

$$|E|^2 = \frac{1}{12}\, R^2 + 4\gamma_1|\eta|^2 - \frac{\delta}{2}\, \Delta R.$$

Multiplying by $|E|$ and integrating by parts gives

$$(5.80) \qquad \int |E|^3 \, dv \;=\; \int \left[\frac{1}{12} R^2 |E| + 4\gamma_1 |\eta|^2 |E| - \frac{\delta}{2} |E| \Delta R \right] dv$$

$$\leq\; \int \left[\frac{1}{12} R^2 |E| + \frac{\delta}{2} \nabla |E| \nabla R \right] dv$$

$$\leq\; \int \left[\frac{1}{12} R^2 |E| + \frac{\delta}{2} |\nabla E|^2 + \frac{\delta}{2} |\nabla R|^2 \right] dv.$$

Using the inequality $R^2 |E| \leq \frac{2}{3} R^3 + \frac{1}{3} |E|^3$, we conclude from (5.80) that

$$(5.81) \qquad \int |E|^3 \, dv \;\lesssim\; \int \left[\delta |\nabla E|^2 + \delta |\nabla R|^2 + R^3 \right] dv.$$

Integrating (5.10) over M^4 we obtain the identity

$$\int |\nabla E|^2 \, dv \;=\; \int \left[\frac{1}{12} |\nabla R|^2 - 2\operatorname{tr} E^3 - \frac{1}{3} R|E|^2 \right.$$

$$\left. + 2W_{ijk\ell} E_{ik} E_{j\ell} + 2E_{ij} B_{ij} \right] dv,$$

so that

$$\int |\nabla E|^2 \, dv \;\leq\; \int \left[\frac{1}{12} |\nabla R|^2 + C|E|^3 + C \right] dv.$$

Substituting this into (5.81) gives

$$\int |E|^3 \, dv \;\lesssim\; \int \left[R^3 + \delta |\nabla R|^2 + \delta |E|^3 + C \right] dv$$

$$\Longrightarrow \int |E|^3 \, dv \;\lesssim\; \int \left[R^3 + \delta |\nabla R|^2 + C \right] dv.$$

Finally, appealing to Lemma 5.9 along with inequality (5.79) we get (5.78). $\square$

Substituting (5.78) into (5.77), we obtain

$$\left(\int |\nabla w|^{12} dv \right)^{\frac{1}{4}} \;\lesssim\; \int \left[R^3 + |\nabla w|^6 + 1 \right] dv.$$

Then (5.72) follows from (5.4). $\square$

Remark. In the remainder of this section we return to our convention of computing in the metric g and suppressing the volume form.

LEMMA 5.24.

(i) $\int |\nabla w|^6 \leq C$,

(ii) $\int |\nabla w|^{12} \leq C$,

(iii) $\int R^3 \leq C$,

(iv) $\int |A|^3 \leq C$,

(v) $\int |\nabla^2 w|^3 \leq C$,

(vi) $\| w \|_{C^\alpha} \leq C$, for $\alpha < \frac{1}{3}$.

Proof. (i) We begin with integration by parts:

$$\int |\nabla w|^6 = \int \nabla w \nabla w |\nabla w|^4 = \int -w \Delta w |\nabla w|^4 - w \nabla w \nabla |\nabla w|^4$$
$$\lesssim \int |w| |\nabla^2 w| |\nabla w|^4.$$

By Hölder's inequality,

$$\int |\nabla w|^6 \lesssim \left(\int |\nabla^2 w|^2 |\nabla w|^2 \right)^{\frac{1}{2}} \left(\int |\nabla w|^6 w^2 \right)^{\frac{1}{2}}$$
$$\leq \left(\int |\nabla^2 w|^2 |\nabla w|^2 \right)^{\frac{1}{2}} \left(\int |\nabla w|^{12} \right)^{\frac{1}{8}} \left(\int |\nabla w|^4 |w|^{\frac{8}{3}} \right)^{\frac{3}{8}}.$$

Appealing to (5.62), (5.72) and (3.4), we obtain

$$\int |\nabla w|^6 \lesssim \left(\int \delta |\nabla w|^6 + R^2 + 1 \right)^{\frac{1}{2}} \left(\int |\nabla w|^6 + 1 \right)^{\frac{1}{2}}$$
$$\lesssim \delta^{\frac{1}{2}} \left(\int |\nabla w|^6 \right) + \left(\int R^2 \right)^{\frac{1}{2}} \left(\int |\nabla w|^6 \right)^{\frac{1}{2}}$$
$$+ \left(\int R^2 \right)^{\frac{1}{2}} + \left(\int |\nabla w|^6 \right)^{\frac{1}{2}} + 1,$$

which implies

(5.82)
$$\int |\nabla w|^6 \lesssim \left(\int R^2 + 1 \right).$$

By (5.4)

$$\left(\int R^2 \right) \lesssim \left(\int R^3 \right)^{\frac{2}{3}} \lesssim \left(\int |\nabla w|^6 \right)^{\frac{2}{3}} + 1.$$

And combining this with (5.82) gives (i).

(ii) This is immediate from (i) and (5.72).

(iii) follows from (i) and (5.4).

(iv) follows from (iii) and (5.78).

(v) follows from (i), (iv), and (5.76).

(vi) Notice that by (3.3),

$$
\begin{aligned}
\int |\nabla_0 w|^6 \, dv_0 \;&=\; \int |\nabla w|^6 \, e^{2w} \, dv \\
&\leq\; \left(\int |\nabla w|^{12} dv \right)^{\frac{1}{2}} \left(\int e^{4w} dv \right)^{\frac{1}{2}} \\
&\lesssim\; \left(\int e^{8w} \, dv_0 \right)^{\frac{1}{2}} \leq C.
\end{aligned}
$$

The result then follows from the Sobolev embedding theorem. □

Proof of Theorem 5.1. Simply apply Lemma 5.24, using (5.74). □

In conclusion, we note an important corollary.

COROLLARY 5.25.

$$
(5.83) \qquad\qquad \int \delta \left(\frac{\Delta R}{R} \right)^2 dv \leq C.
$$

Proof. From (5.61) and Lemma 5.24 we conclude that

$$
\int \delta \frac{(\Delta R)^2}{R} \leq C.
$$

Therefore,

$$
(5.84) \qquad\qquad \int \delta \left(\frac{\Delta R}{R} \right)^2 \leq \left(\frac{1}{\min R} \right) \int \delta \frac{(\Delta R)^2}{R}.
$$

Then (5.83) follows from (5.84) and the following result:

LEMMA 5.26.
$$
\min R \geq C_0 > 0.
$$

Proof. By $(*)_\delta$,

$$
\delta \Delta R \;=\; 8\gamma_1 |\eta|^2 - 2|E|^2 + \frac{1}{6} R^2 \;\leq\; 8\gamma_1 |\eta|^2 + \frac{1}{6} R^2.
$$

Evaluating at the minimum point of R and appealing to Lemma 5.24 (vi), we have

$$
\begin{aligned}
(\min R)^2 \;&\geq\; -48\gamma_1 \, \min |\eta|^2 \\
&=\; -48\gamma_1 \, \min e^{-4w} |\eta|_0^2 \\
&\geq\; 48(-\gamma_1) \, (\max e^{4w})^{-1} \, (\min |\eta|_0^2) \\
&\geq\; C_0 > 0.
\end{aligned}
$$
□

$$\textbf{6. } \textit{A priori } W^{2,s} \textbf{ estimates for } s < 5$$

This section will be an extension of the *a priori* estimates of Section 5. Our goal is to modify the argument to establish the following:

THEOREM 6.1.   *Let* $g = e^{2w} g_0$ *be a solution of* $(*)_\delta$ *with positive scalar curvature, normalized so that* $\int w \, dv_0 = 0$. *Assume*

$$(6.1) \qquad \int \sigma_2(A_0) \, dv_0 = \int \sigma_2(A) \, dv > 0.$$

Then there are constants $C_s = C(g_0, s)$ *and* $\delta_0 < 1$ *such that*

$$(6.2) \qquad \int |\nabla_0^2 w|^s \, dv_0 \leq C_s$$

for any $0 < s < 5$ *and* $0 < \delta \leq \delta_0$.

As a direct corollary of the Sobolev embedding theorem, we have the following $\mathcal{C}^{1,\alpha}$ *a priori* bound for the solution w of $(*)_\delta$.

COROLLARY 6.2.   *Under the assumptions of Theorem* 6.1, *there is a constant* $C_\alpha = C(g_0, \alpha)$, *so that*

$$(6.3) \qquad \| w \|_{\mathcal{C}^{1,\alpha}} \leq C_\alpha \text{ for all } \alpha < \frac{1}{5} \text{ and } \delta \leq \delta_0.$$

The proof of (6.2) follows the same pattern as the proof of (5.2) in Section 5, with the exception that the terms contributed by $\delta \Delta R$ are more complicated than before and need to be handled with more care.

To start the proof, in analogy with (5.8) and (5.31), for each $0 \leq p \leq 2$, define

$$(6.4) \qquad \mathrm{I}^p = \int S_{ij} \, \nabla_i \nabla_j \, R^{p+1},$$

$$(6.5) \qquad \mathrm{II}^p = \int S_{ij} \, \nabla_i (R^p \nabla_j V),$$

where $V = \frac{1}{2} |\nabla w|^2$. Since S is divergence-free (see (5.5)), both I^p and $\mathrm{II}^p = 0$. Thus our strategy is to show that some combination of the terms I^p and II^p is bounded below by a multiple of $\int R^{p+3}$ plus some lower order terms.

We begin by splitting the terms I^p, II^p as follows:

$$(6.6) \quad \mathrm{I}^p = \int S_{ij} \, \nabla_i \nabla_j \, R^{p+1} = \int S_{ij} \, \nabla_i \left((p+1) \, \nabla_j R \, R^p \right)$$

$$= (p+1) \int R^p \, S_{ij} \, \nabla_i \nabla_j R + p(p+1) \int R^{p-1} \, S_{ij} \nabla_i R \, \nabla_j R$$

$$= \mathrm{I}_1^p + \mathrm{I}_2^p;$$

and

$$\text{(6.7)} \qquad \text{II}^p \;=\; \int S_{ij} \,\nabla_i (R^p \,\nabla_j V)$$

$$=\; p \int R^{p-1}\, S_{ij}\, \nabla_i R \,\nabla_j V \;+\; \int R^p\, S_{ij}\, \nabla_i \nabla_j V$$

$$=\; \text{II}_1^p \;+\; \text{II}_2^p.$$

Estimating I_1^p. We now apply the identity (5.10), the estimate (5.15), and argue as in the proof of Lemma 5.5 to obtain

$$\text{I}_1^p \;=\; (p+1) \int R^p\, S_{ij}\, \nabla_i \nabla_j R$$

$$=\; (p+1) \int R^p \left[3\Delta\sigma_2(A) \,+\, 3(|\nabla E|^2 - \frac{1}{12}(\nabla R|^2) \right.$$

$$\left. +\, 6\mathrm{tr}\, E^3 \,+\, R|E|^2 \,-\, 6W_{ikj\ell}\, E_{ij}\, E_{k\ell} \,-\, 6B_{ij}\, E_{ij} \right]$$

$$\geq\; 3(p+1) \int \Delta(R^p)\, \sigma_2(A)$$

$$+\, (p+1) \int \left[\frac{3}{2}\delta\, R^{p-1}\, (\Delta R)^2 \,+\, \frac{3}{2}\delta p\, R^{p-2} |\nabla R|^2\, \Delta R \right.$$

$$\left. +\, 12\gamma_1\, R^{p-1} <\nabla R, \nabla|\eta|^2> \,-\, 12\gamma_1\, R^{p-2}\, |\eta|^2\, |\nabla R|^2 \right]$$

$$+\, (p+1) \int (6R^p\, \mathrm{tr}\, E^3 \,+\, R^{p+1}\, |E|^2) \,-\, C \int R^{p+2} \,-\, C.$$

Using $(*)_\delta$ we obtain

$$\text{(6.8)} \quad \text{I}_1^p \;\geq\; 3(p+1) \int \Delta(R^p) \left[\frac{\delta}{4}\Delta R - 2\gamma_1 |\eta|^2 \right]$$

$$+\, (p+1) \int \left[\frac{3}{2}\delta\, R^{p-1}\, (\Delta R)^2 \,+\, \frac{3}{2}\delta p\, R^{p-2}\, |\nabla R|^2\, \Delta R \right.$$

$$\left. +\, 12\gamma_1\, R^{p-1}\, \langle\nabla R, \nabla|\eta|^2\rangle \,-\, 12\gamma_1\, R^{p-2}\, |\eta|^2\, |\nabla R|^2 \right]$$

$$+\, (p+1) \int \left(6R^p\, \mathrm{tr}\, E^3 \,+\, R^{p+1}\, |E|^2 \right) \,-\, C \int R^{p+2} \,-\, C$$

$$\geq\; \text{I}_{1,\delta}^p \,+\, 3(p+1) \int -2\gamma_1 \Delta(R^p)|\eta|^2$$

$$+3(p+1) \int \left[4\gamma_1 R^{p-1} <\nabla R, \nabla|\eta|^2> \,-\, 4\gamma_1 R^{p-2}|\eta|^2|\nabla R|^2 \right]$$

$$+\, (p+1) \int \left(6R^p\, \mathrm{tr}\, E^3 \,+\, R^{p+1}\, |E|^2 \right) \,-\, C \int R^{p+2} \,-\, C,$$

where

$$\text{(6.9)} \qquad \text{I}_{1,\delta}^p \;=\; \frac{3}{4}\delta(p+1) \int \left[\Delta(R^p)\, (\Delta R) \,+\, 2R^{p-1}\, (\Delta R)^2 \right.$$

$$\left. +\, 2p\, R^{p-2}\, |\nabla R|^2\, \Delta R \right].$$

We can estimate the terms involving η in (6.8) by integrating by parts, using the Schwartz inequality, and the fact that $\gamma_1 < 0$, as follows:

$$3(p+1) \int \left[-2\gamma_1 \Delta(R^p)|\eta|^2 + 4\gamma_1 R^{p-1} < \nabla R, \nabla|\eta|^2 > -4\gamma_1 R^{p-2}|\eta|^2|\nabla R|^2 \right]$$

$$= 3(p+1) \int \left[2\gamma_1(p+2)R^{p-1} < \nabla R, \nabla|\eta|^2 > -4\gamma_1 R^{p-2}|\eta|^2|\nabla R|^2 \right]$$

$$\geq -C \int R^p |\nabla|\eta||^2.$$

Since $|\nabla|\eta||^2 = |\nabla(e^{-2w}|\eta|_0)|^2$ and $p \leq 2$, we can use the results of Lemma 5.24 (i.e., $||\nabla w||_{12} \leq C, \ w \geq -C$) to conclude $-C \int R^p|\nabla|\eta||^2 \geq -C \int R^{p+2} - C$. Substituting this into (6.8) we have

(6.10) $$\qquad I_1^p \geq I_{1,\delta}^p + (p+1) \int (6R^p \mathrm{tr}\, E^3 + R^{p+1}|E|^2) - C \int R^{p+2} - C.$$

We now introduce the notation:

(6.11) $$\qquad\qquad A_p \ = \ \int R^{p-1}(\Delta R)^2,$$

$$B_p \ = \ \int R^{p-2}|\nabla R|^2 \Delta R.$$

With this notation, we may rewrite (6.9) as

(6.12) $$\qquad\qquad I_{1,\delta}^p = \frac{3}{4}\delta(p+1)\left[(p+2)A_p + p(p+1)B_p\right].$$

The following material is fairly technical, but the overall goal is to establish (6.31) below:

$$I_{1,\delta}^p \ + \ I_2^p \ + \ 24(p+1)\, II_1^p \ \geq \ C\delta(A_p + C_p) - C\left(\int R^{p+3}\right)^{\frac{p+2}{p+3}} - C,$$

for any $\delta > 0, p \leq 2$. This will require additional notation as well. We begin with:

LEMMA 6.3.   *Denote* $\overset{\circ}{\nabla}{}^2 R = \nabla^2 R - \frac{1}{4}\Delta R\, g_{ij}$, *and*

$$C_p \ = \ \int R^{p-3}|\nabla R|^4,$$

$$\overset{\circ}{A}_p \ = \ \int R^{p-1}|\overset{\circ}{\nabla}{}^2 R|^2,$$

$$D_p \ = \ \int R^{p-2}\nabla_i\nabla_j R\,\nabla_i R\,\nabla_j R,$$

$$\overset{\circ}{D}_p \ = \ \int R^{p-2}\overset{\circ}{\nabla}{}^2_{ij} R\,\nabla_i R\,\nabla_j R.$$

Then

$$(6.13) \qquad 3pB_p \;=\; 4\overset{\circ}{A}_p - 3A_p - 2(p-2)\,C_p + 4(p-2)\,\overset{\circ}{D}_p.$$
$$+ 4\int \mathrm{Ric}\,(\nabla R, \nabla R)\,R^{p-1}.$$

Proof. Recall the Bochner identity:

$$(6.14) \qquad \frac{1}{2}\Delta\,|\nabla R|^2 \;=\; |\nabla^2 R|^2 + \mathrm{Ric}(\nabla R, \nabla R) + \langle \nabla R, \nabla\Delta R\rangle.$$

Then integration by parts along with (6.14) give

$$(6.15) \quad B_p \;=\; \int R^{p-2}|\nabla R|^2\,\Delta R$$
$$=\; \int \Delta\left(R^{p-2}\,|\nabla R|^2\right) R$$
$$=\; \int \Delta\left(R^{p-2}\right)|\nabla R|^2\,R + 2\int \langle \nabla R^{p-2},\, \nabla|\nabla R|^2\rangle R$$
$$+ \int \Delta|\nabla R|^2\,R^{p-1}$$
$$=\; (p-2)\int R^{p-2}\,|\nabla R|^2\,\Delta R + (p-2)(p-3)\int R^{p-3}\,|\nabla R|^4$$
$$+ 4(p-2)\int R^{p-2}\,\nabla_i\nabla_j R\,\nabla_i R\,\nabla_j R + 2\int R^{p-1}\,|\nabla^2 R|^2$$
$$+ 2\int R^{p-1}\,\mathrm{Ric}(\nabla R, \nabla R) + 2\int R^{p-1}\langle \nabla R,\,\nabla(\Delta R)\rangle.$$

Rewriting the last term in (6.15) and integrating by parts once again, we obtain

$$(6.16) \quad \int R^{p-1}\,\langle \nabla R,\,\nabla(\Delta R)\rangle \;=\; \frac{1}{p}\int \nabla R^p\,\nabla(\Delta R) = -\frac{1}{p}\int \Delta\,R^p\,\Delta R$$
$$=\; -\frac{1}{p}\,(p\,A_p + p(p-1)B_p)$$
$$=\; -(A_p + (p-1)B_p).$$

Substituting (6.16) into (6.15), we obtain

$$(6.17) \quad B_p \;=\; -2A_p - pB_p + (p-2)\,(p-3)\,C_p$$
$$+ 4(p-2)\,D_p + 2\int R^{p-1}|\nabla^2 R|^2 + 2\int R^{p-1}\,\mathrm{Ric}(\nabla R, \nabla R).$$

There are two ways to express the term D_p. First, we can write

$$(6.18) \qquad D_p \;=\; \int R^{p-2}\left(\nabla_i\nabla_j R - \frac{1}{4}\Delta R g_{ij}\right)\nabla_i R\,\nabla_j R$$
$$+ \frac{1}{4}\int R^{p-2}\,|\nabla R|^2\,\Delta R$$
$$=\; \overset{\circ}{D}_p + \frac{1}{4}\,B_p,$$

and substituting (6.18) into (6.17), we get

$$(6.19) \quad 3B_p = -\frac{3}{2} A_p + 2 \overset{\circ}{A}_p + 4(p-2) \overset{\circ}{D}_p + (p-2)(p-3) C_p$$
$$+ 2 \int R^{p-1} \operatorname{Ric}(\nabla R, \nabla R).$$

Alternatively, we can integrate by parts and express D_p as

$$(6.20) \quad D_p = \frac{1}{2} \int R^{p-2} \nabla_i R \nabla_i |\nabla R|^2 = -\frac{1}{2} (B_p + (p-2) C_p).$$

Substituting (6.20) back into (6.17), we obtain

$$(6.21) \quad 3(p-1) B_p = -\frac{3}{2} A_p + 2 \overset{\circ}{A}_p - (p-1)(p-2) C_p + 2 \int R^{p-1} \operatorname{Ric}(\nabla R, \nabla R).$$

Summing (6.19) and (6.21), we obtain the identity (6.13) in the lemma. $\qquad \square$

COROLLARY 6.4. *For $p < 2$,*

$$(6.22) \quad 3(p-\delta) B_p \geq -3 A_p + C_p(2-p) \left(\frac{1}{2} + \frac{3}{4} p\right).$$

Proof. Applying the sharp inequality of [SW, p. 234], we have

$$(6.23) \quad |\overset{\circ}{\nabla}^2 R(\nabla R, \nabla R)| \leq \frac{\sqrt{3}}{2} |\overset{\circ}{\nabla}^2 R| |\nabla R|^2.$$

Therefore,

$$(6.24) \quad 4(2-p) |\overset{\circ}{D}_p| \leq 2\sqrt{3}(2-p) \overset{\circ}{A}_p^{1/2} C_p^{1/2}$$
$$\leq 4 \overset{\circ}{A}_p + \frac{3}{4}(2-p)^2 C_p.$$

Substituting (6.24) into (6.13), then applying the inequality $\operatorname{Ric}(\nabla R, \nabla R)$ $\geq \frac{3\sigma_2(A)}{R}|\nabla R|^2$, we obtain (6.22). $\qquad \square$

We will now begin to estimate II_1^p. Our strategy is to establish that

$$\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 24(p+1)\mathrm{II}_1^p \gtrsim \delta(A_p + C_p) + \text{ lower order terms}$$

as in (6.31).

LEMMA 6.5. *There is a constant $C = C(g_0)$ such that for any $\varepsilon > 0$, $\eta > 0$,*

$$(6.25) \quad \mathrm{II}_1^p \geq -\frac{1}{2} p\varepsilon^2 \int R^{p-1} S_{ij} \nabla_i R \nabla_j R - C \delta\varepsilon^2 \eta A_p - C\delta\varepsilon^2 \eta^{-1} C_p$$
$$- C \varepsilon^{-6} \eta^{-1} \left(\int R^{p+3}\right)^{\frac{p+1}{p+3}} - Cp\varepsilon^{-2} \left(\int R^{p+3}\right)^{\frac{p+2}{p+3}}.$$

Proof. For any $\varepsilon > 0$, we can write II_1^p as

(6.26)

$$
\begin{aligned}
\mathrm{II}_1^p &= p \int R^{p-1} S_{ij} \nabla_i R \nabla_j V \\
&= \frac{1}{2} p \int R^{p-1} S_{ij} \nabla_i \left(\varepsilon R + \frac{1}{\varepsilon} V \right) \nabla_j \left(\varepsilon R + \frac{1}{\varepsilon} V \right) \\
&\quad - \frac{1}{2} p \varepsilon^2 \int R^{p-1} S_{ij} \nabla_i R \nabla_j R - \frac{1}{2} p \varepsilon^{-2} \int R^{p-1} S_{ij} \nabla_i V \nabla_j V.
\end{aligned}
$$

We notice that for each $\eta > 0$,

$$
(6.27) \quad \int R^{p-1} S_{ij} \nabla_i \left(\varepsilon R + \frac{1}{\varepsilon} V \right) \nabla_j \left(\varepsilon R + \frac{1}{\varepsilon} V \right)
$$

$$
\geq \frac{3}{4} \delta \int R^{p-1} \frac{\Delta R}{R} \left| \nabla \left(\varepsilon R + \frac{1}{\varepsilon} V \right) \right|^2
$$

$$
\geq -\frac{3}{2} \delta \left(\int R^{p-1} (\Delta R)^2 \right)^{1/2} \left(\int R^{p-3} |\nabla (\varepsilon R + \frac{1}{\varepsilon} V|^4 \right)^{1/2}
$$

$$
\geq -C \delta A_p^{1/2} \left[\varepsilon^2 C_p^{1/2} + \varepsilon^{-2} \left(\int R^{p-3} |\nabla V|^4 \right)^{1/2} \right]
$$

$$
\geq -C \delta \varepsilon^2 \eta A_p - C \delta \varepsilon^2 \eta^{-1} C_p - C \varepsilon^{-6} \eta^{-1} \int R^{p-3} |\nabla V|^4 - C.
$$

We now estimate the term $\int R^{p-3} |\nabla V|^4$. Since $V = \frac{1}{2} |\nabla w|^2$, we have $\nabla_i V = \nabla_i \nabla_j w \nabla_j w$, and $|\nabla V| \lesssim |\nabla^2 w| |\nabla w|$. Thus

$$
(6.28) \qquad\qquad \int R^{p-3} |\nabla V|^4 \ \lesssim \ \int R^{p-3} |\nabla^2 w|^4 |\nabla w|^4
$$

and

$$
(6.29) \qquad \int R^{p-1} S_{ij} \nabla_i V \nabla_j V \ \lesssim \ \int R^{p-1} |\nabla^2 w|^3 |\nabla w|^2
$$

$$
+ \int R^{p-1} |\nabla^2 w|^2 |\nabla w|^4.
$$

Substituting (6.28) into (6.27), then substituting (6.27), (6.29) into (6.26), we see that inequality (6.25) in Lemma 6.5 is a direct consequence of the following technical lemma.

LEMMA 6.6. *Suppose w satisfies (5.2) and (5.3). Then there is a constant $C = C(g_0)$ such that for any nonnegative positive number a, b, c with $s = a + b + \frac{c}{2} \leq 6$,*

$$
(6.30) \qquad\qquad \int R^a |\nabla^2 w|^b |\nabla w|^c \ \lesssim \ \left(\int R^s \right)^{\frac{a+b}{s}} + C.
$$

Proof. Since $|R| \lesssim |R_0| + |\Delta w| + |\nabla w|^2$, by Holder's inequality

$$\int R^a\,|\nabla^2 w|^b |\nabla w|^c$$

$$\lesssim \int |\nabla^2 w|^{a+b} |\nabla w|^c + \int |\nabla^2 w|^b\,|\nabla w|^{2a+c}$$

$$+ \int |\nabla^2 w|^b\,|\nabla w|^c$$

$$\lesssim \left(\int |\nabla^2 w|^s \right)^{\frac{a+b}{s}} \left(\int |\nabla w|^{2s} \right)^{\frac{c}{2s}} + \left(\int |\nabla^2 w|^s \right)^{\frac{b}{s}} \left(\int |\nabla w|^{2s} \right)^{\frac{2a+c}{2s}}$$

$$+ \left(\int |\nabla^2 w|^s \right)^{\frac{b}{s}} \left(\int |\nabla w|^{\frac{2cs}{2a+c}} \right)^{\frac{2a+c}{2s}}.$$

By (5.3), for $s \le 6$, we have $2s \le 12$ and $\nabla w \in L^{12}$; hence

$$\int R^a\,|\nabla^2 w|^b |\nabla w|^c \lesssim \left(\int |\nabla^2 w|^s \right)^{\frac{a+b}{s}} + C.$$

To finish the proof of (6.30), we simply observe that since w satisfies (5.2) (i.e. w is bounded) and $s \le 6$, by elliptic regularity,

$$\int |\nabla^2 w|^s \lesssim \int \left(|\nabla_0^2 w|^s + |\nabla_0 w|^{2s} \right) dv_0 \lesssim \int |\nabla_0^2 w|^s\, dv_0 + C$$

$$\lesssim \int (\Delta_0 w)^s\, dv_0 + C \lesssim \int R^s\, dv_0 + C. \qquad \square$$

LEMMA 6.7. *There is a constant C such that for each $\delta > 0$, $p < 2$,*

$$(6.31) \qquad \mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 24(p+1)\,\mathrm{II}_1^p \ge C\delta(A_p + C_p) - C\left(\int R^{p+3} \right)^{\frac{p+2}{p+3}} - C.$$

Proof. Combining (6.6), (6.12) and (6.25), we have for each $\varepsilon > 0$ small enough so $12\varepsilon^2 < p+1$, $\eta > 0$,

$$\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 24(p+1)\,\mathrm{II}_1^p$$

$$\ge \frac{3}{4}\,\delta(p+1)\,[(p+2)\,A_p + p(p+1)B_p]$$

$$+ \left(p(p+1) - 12p(p+1)\varepsilon^2 \right) \int R^{p-1}\,S_{ij}\,\nabla_i R\,\nabla_j R - C\delta\varepsilon^2\eta\,A_p$$

$$- C\delta\varepsilon^2\,\eta^{-1}\,C_p - C\varepsilon^{-6}\eta^{-1} \left(\int R^{p+3} \right)^{\frac{p+1}{p+3}} - Cp\varepsilon^{-2} \left(\int R^{p+3} \right)^{\frac{p+2}{p+3}}.$$

By the fact that

$$\int R^{p-1} S_{ij}\nabla_i R \nabla_j R \ge \int 3R^{p-2}\sigma_2(A)|\nabla R|^2$$

$$= \int 3R^{p-2}|\nabla R|^2(\frac{\delta}{4}\Delta R - 2\gamma_1|\eta|^2)$$

$$\ge \frac{3}{4}\delta B_p,$$

the preceding estimate becomes:

(6.32)

$$
\begin{aligned}
\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p &+ 24(p+1)\,\mathrm{II}_1^p \\
&\geq \frac{3}{4}\,\delta(p+1)\left[(p+2)\,A_p + p(p+1)\,B_p + (p-12p\varepsilon^2)\,B_p\right] - C\delta\varepsilon^2\,\eta\,A_p \\
&\quad - C\delta\varepsilon^2\,\eta^{-1}\,C_p - C\varepsilon^{-6}\,\eta^{-1}\left(\int R^{p+3}\right)^{\frac{p+1}{p+3}} - Cp\varepsilon^{-2}\left(\int R^{p+3}\right)^{\frac{p+2}{p+3}}.
\end{aligned}
$$

Thus if $\delta < 1 \leq p < 2$, we may apply (6.22) and (6.32) to obtain that for all $\eta > 0$,

(6.33)

$$
\begin{aligned}
\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p &+ 24(p+1)\,\mathrm{II}_1^p \\
&\geq 9\delta\varepsilon^2\,(p+1)\,A_p + a_p\,\delta C_p - a_p\delta^2 A_p - C\delta\varepsilon^2\,\eta\,A_p \\
&\quad - C\delta\varepsilon^2\,\eta^{-1}C_p - C\varepsilon^{-6}\,\eta^{-1}\left(\int R^{p+3}\right)^{\frac{p+1}{p+3}} - Cp\varepsilon^{-2}\left(\int R^{p+3}\right)^{\frac{p+2}{p+3}},
\end{aligned}
$$

where a_p is a positive constant depending only on p.

Thus if we first choose η small enough so that $C\eta < 8(p+1)$, and then choose ε sufficiently small so that $a_p > C\varepsilon^2\eta^{-1}$, then for δ sufficiently small we conclude from (6.33) that (6.31) holds. $\qquad\square$

We will now estimate the term $\mathrm{II}_2^p = \int R^p S_{ij}\nabla_i\nabla_j V$.

PROPOSITION 6.8. *There is a constant C such that for $p < 2$, for each $\gamma > 0$,*

$$
\begin{aligned}
(6.34)\quad \mathrm{II}_2^p \;\geq\; &\int R^p\left(-\frac{1}{4}\operatorname{tr}E^3 + \frac{1}{288}R^3\right) \\
&- C\gamma\delta A_p - C\gamma\delta C_p - C\gamma^{-1}\delta\int R^{p+3} - C\int R^{p+2} - C.
\end{aligned}
$$

Proof. The proof of this proposition follows the pattern of the proof of Proposition 5.18. However, the estimates are less delicate because we already know $w \in L^\infty$ and $\nabla w \in L^{12}$ in view of Theorem 5.1. We will outline the proof but skip some of the details.

To begin with, we have from (5.45) in Proposition 5.16,

$$
\begin{aligned}
(6.35)\quad R^p\,S_{ij}\,\nabla_i\nabla_j V \;\geq\; &R^p\left(-\frac{1}{4}\operatorname{tr}E^3 + \frac{1}{48}R|E|^2 + \frac{1}{576}R^3\right) \\
&- \frac{1}{2}\,R^p\,\langle\nabla w, \nabla\sigma_2(A)\rangle \\
&- \frac{1}{4}\,R^{p+1}\,|\nabla w|^4 - R^p\,S_{ij}\,\nabla_i|\nabla w|^2\,\nabla_j w \\
&- C\,R^p|\mathrm{Ric}|^2 - C\,R^p|\mathrm{Ric}|\,|\nabla w|^2 - CR^p.
\end{aligned}
$$

By $(*)_\delta$,

$$(6.36) \qquad \int R^{p+1} |E|^2 = \int R^{p+1} \left(\frac{1}{12} R^2 - \frac{\delta}{2} \Delta R + 4\gamma_1 |\eta|^2 \right)$$

$$\geq \frac{1}{12} \int R^{p+3} - C \int R^{p+1},$$

$$\int R^p \langle \nabla w, \nabla \sigma_2(A) \rangle = - \int R^p \Delta w \sigma_2(A) - \int \nabla (R^p) \nabla w \, \sigma_2(A),$$

so that

$$(6.37) \qquad \left| \int R^p \langle \nabla w \, \nabla \sigma_2(A) \rangle \right|$$

$$\lesssim \left| \int R^p \Delta w \delta \Delta R \right| + \left| \int R^p \Delta w \, |\eta|^2 \right|$$

$$+ \int R^{p-1} |\nabla R| \, \delta |\Delta R| \, |\nabla w| + 2\gamma_1 \int \nabla (R^p) \nabla w |\eta|^2$$

$$\lesssim \delta A_p^{1/2} \left(\int R^{p+1} (\Delta w)^2 \right)^{1/2} + \int R^p |\Delta w|$$

$$+ \delta A_p^{1/2} C_p^{1/4} \left(\int R^{p+1} |\nabla w|^4 \right)^{1/4} - 2\gamma_1 \int R^p \left[\Delta w + \langle \nabla w, \nabla |\eta|^2 \rangle \right].$$

Now, for $p < 2$,

$$(6.38)$$

$$\int R^{p+1} (\Delta w)^2 \lesssim \int R^{p+3} + \int R^{p+1} |\nabla w|^4 + \int R^{p+1}$$

$$\lesssim \int R^{p+3} + \left(\int R^{p+3} \right)^{\frac{p+1}{p+3}} \left(\int |\nabla w|^{2(p+3)} \right)^{\frac{2}{p+3}} + \int R^{p+1}$$

$$\lesssim \int R^{p+3} + C.$$

Applying a similar argument as (6.38) to each of the terms in (6.37), we obtain

$$(6.39) \qquad \left| \int R^p \nabla w \, \nabla \sigma_2(A) \right| \leq \gamma \delta A_p + \gamma^{-1} \delta \int R^{p+3} + \gamma \delta C_p + C$$

for any $\gamma > 0$.

We now observe that integration of the rest of the terms on the right-hand side of (6.35) can be estimated similarly to the corresponding terms in Proposition 5.18.

Combining (6.36), (6.37), (6.40) and our observation above, we obtain the desired estimate (6.34) in Proposition 6.8. $\qquad \square$

Proof of Theorem 6.1. As explained before, our strategy of proof is the same as the strategy of proof of Theorem 5.1. That is, we add up $I^p + 24(p + 1)II^p$, so that the coefficient of the term $\int R^p \text{tr} \, E^3$ in the sum

becomes zero and the rest of the terms in the sum are dominated from below by $\int R^{p+3}$. To be more precise, first we combine (6.6), (6.7), (6.8) and (6.34) to obtain

$$
\begin{aligned}
(6.40) \quad 0 &= \mathrm{I}^p + 24(p+1)\,\mathrm{II}^p \\
&= \mathrm{I}_1^p + \mathrm{I}_2^p + 24(p+1)\,\mathrm{II}_1^p + 24(p+1)\,\mathrm{II}_2^p \\
&\geqslant \mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 24(p+1)\,\mathrm{II}_1^p \\
&\quad + (p+1)\left(\int 6R^p\,\mathrm{tr}\,E^3 + \frac{1}{12}\,R^{p+3}\right) \\
&\quad + 24(p+1)\left(\int -\frac{1}{4}\,R^p\,\mathrm{tr}\,E^3 + \frac{1}{288}\,R^{p+3}\right) - C\,\gamma\delta\,A_p \\
&\quad - C\gamma\delta\,C_p - C\gamma^{-1}\delta\int R^{p+3} - C\int R^{p+2} - C.
\end{aligned}
$$

We then apply (6.31), Lemma 6.7 to estimate the term $\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 24(p+1)\mathrm{II}_1^p$ in (6.40) above. We now choose γ small enough so that in the combined expression the coefficients of the δA_p and δC_p terms are positive. Thus we conclude that there is a constant $C = C(g_0, p)$, so that for all $p < 2$,

$$
\begin{aligned}
(6.41) \quad 0 = \mathrm{I}^p + 24(p+1)\mathrm{II}^p &\geqslant \frac{1}{6}(p+1)\int R^{p+3} - C\delta\int R^{p+3} \\
&\quad - C\left(\int R^{p+3}\right)^{\frac{p+2}{p+3}} - C\int R^{p+2} - C.
\end{aligned}
$$

It follows from (6.41) that for δ sufficiently small and $p < 2$ there is some constant $C = C(g_0, p)$ so that $\int R^{p+3} \leq C$. Thus $\int |\Delta w|^{p+3} \leq C$; from this and the fact that $w \in L^\infty$, $\nabla w \in L^{12}$, we conclude that (6.2) holds. $\qquad\square$

7. Smoothing via the Yamabe flow

In light of our estimates in Sections 4–6, we now have *a priori* $C^{1,\alpha}$ bounds for solutions of $(*)_\delta$ with positive scalar curvature. However, for technical reasons we seem to be unable to improve on this. For example, the integral estimates of Section 6 break down when we attempt to establish an L^p-bound for the scalar curvature as soon as $p \geqslant 5$. In this section we show that once $p > 4$, we can use the Yamabe flow to smooth solutions of $(*)_\delta$ and obtain metrics with $\sigma_2(A) > 0$.

THEOREM 7.1.   *Let $g = e^{2w}g_0$ be a solution of $(*)_\delta$ with positive scalar curvature, normalized so that $\int w\,dv_0 = 0$. Assume $\int \sigma_2(A)\,dv > 0$. If δ is sufficiently small, then there is a smooth conformal metric $h = e^{2v}g$ such that $\sigma_2(A_h) > 0$.*

The proof of Theorem 7.1 is based on estimates for solutions of the Yamabe flow using parabolic Moser iteration. The nonlinear nature of the flow obviously complicates matters, but we will see that our cause is aided by the fact that the evolution of the quantity $f \equiv \sigma_2(A) + 2\gamma_1|\eta|^2$ is fairly simple to analyze (more precisely, we will study the quantity $\frac{f}{R}$; see (7.24)).

A curious feature of the analysis in this section is the necessity of *a priori* L^p bounds for the curvature of the initial data with $p > 4$. Typically, the smoothing effects of semi-linear heat flows, like the Yamabe or Ricci flows, only require $p > \frac{n}{2} = 2$. But to obtain in addition a positive lower bound for $\sigma_2(A)$ we actually need $p > 4$; see the proof of Theorem 7.1 at the end of the section.

We begin with a basic short-time existence result, based on the work of [Ha], [Ye].

PROPOSITION 7.2.   *Let g satisfy the hypotheses of Theorem 7.1. Consider*

(7.1)
$$\begin{cases} \frac{\partial h}{\partial t} = -\frac{1}{3} Rh, \\ h(0, \cdot) = g = e^{2w}g_0. \end{cases}$$

Then there exists a $T_0 = T_0(g_0)$ such that (7.1) has a unique smooth solution for $t \in [0, T_0)$.

Proof. On a compact n-dimensional Riemannian manifold, consider the normalized Yamabe flow

(7.1′)
$$\begin{cases} \frac{\partial h}{\partial t} = -\frac{1}{(n-1)} (R - r)h, \\ r(t) = \int R\, dv / \int dv, \\ h(0, \cdot) = h_0. \end{cases}$$

Then (7.1)′ is known to admit a unique smooth solution for all time (see [Ha], [Ye]). When $n = 4$, (7.1) and (7.1)′ differ only by a rescaling in time and space in order to normalize the volume. Therefore, our result will follow if we can produce a time interval (depending on g_0 alone) on which the volume of h is controlled.

To this end, let us record some basic consequences of (7.1).

LEMMA 7.3 (See, for example, [Ch]).   *Under (7.1),*

(7.2)
$$\frac{\partial}{\partial t} dv = -\frac{2}{3} R dv,$$

(7.3)
$$\frac{\partial R}{\partial t} = \Delta R + \frac{1}{3} R^2.$$

Remark. Since the initial metric $h(0, \cdot) = g$ has positive scalar curvature, it follows from applying the minimum principle to (7.3) that the scalar curvature

of h remains positive for as long as the solution exists. Indeed, by Lemma 5.26 the scalar curvature must satisfy

$$(7.4) \qquad\qquad R \geq C(g_0) > 0.$$

From (7.2) we see that the volume is decreasing:

$$\frac{d}{dt} \int dv = \int -\frac{2}{3} R dv < 0.$$

Also,

$$(7.5) \qquad\qquad \frac{d}{dt} \int dv \geq -\frac{2}{3} \left(\int R^2 dv \right)^{\frac{1}{2}} \left(\int dv \right)^{\frac{1}{2}}$$

$$\Longrightarrow \frac{d}{dt} \left(\int dv \right)^{\frac{1}{2}} \geq -\frac{1}{3} \left(\int R^2 dv \right)^{\frac{1}{2}}.$$

By (7.3),

$$(7.6) \quad \frac{d}{dt} \int R^2 dv = \int 2R \left(\Delta R + \frac{1}{3} R^2 \right) dv + R^2 \left(-\frac{2}{3} R dv \right)$$

$$= \int -2|\nabla R|^2 dv \leq 0.$$

From (7.5) and (7.6) we conclude that

$$\left[\mathrm{vol}\, (h(0, \cdot))^{\frac{1}{2}} - C_0 t \right]^2 \leq \mathrm{vol}\, h(t) \leq \mathrm{vol}\, (h(0, \cdot))$$

where $C_0 = C_0(\| R_g \|_{L^2})$. By Lemma 5.24, $\| R_g \|_{L^2} \leq C(g_0)$, and this completes the proof. $\qquad\qquad\square$

PROPOSITION 7.4.   *Let g satisfy the hypotheses of Theorem 7.1. Fix $s \in (4, 5)$. Then there is a $T_1 = T_1(g_0) < T_0$ such that for $t \leq T_1$, the solution $h = e^{2v} g$ of (7.1) satisfies*

(i) $\| \mathrm{Ric}_h \|_{L^s} \leq 2 \| \mathrm{Ric}_g \|_{L^s}$;

(ii) $\| \mathrm{Ric}_h \|_\infty \leq C_2 t^{-\frac{2}{s}}$, *where $C_2 = C_2(g_0)$;*

(iii) $\| v \|_\infty \leq C(g_0)$.

Proof. The proof of Proposition 7.4 relies on estimates for the Ricci flow derived in [Ya], summarized in the following lemma:

LEMMA 7.5 (See [Ya]).   *Assume that with respect to the metric $h = h(t)$, $0 \leq t \leq T$, the following Sobolev inequality holds:*

$$(7.7) \quad \left(\int |\varphi|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq C_S \left[\int |\nabla \varphi|^2 dv + \int \varphi^2 dv \right], \varphi \in W^{1,2}(M^n).$$

Also, let b be a nonnegative function on $M^n \times [0, T]$ such that

$$\frac{\partial}{\partial t} \, dv \leq b \, dv.$$

Let $q > n$, and suppose $u \geq 0$ is a function on $M^n \times [0, T]$ satisfying

$$\frac{\partial u}{\partial t} \leq \Delta u + bu,$$

and that

$$\sup_{0 \leq t \leq T} \| b \|_{L^{q/2}} \leq \beta.$$

Given $p_0 > 1$, there exists a constant $C = C(n, q, p_0, C_S, \beta)$ such that for $0 \leq t \leq T$,

$$(7.8) \qquad \| u(t, \cdot) \|_\infty \leq C e^{Ct} t^{-\frac{n}{2p_0}} \| u(0, \cdot) \|_{p_0} .$$

Moreover, given $p \geq p_0 > 1$, the following inequality holds for $0 \leq t \leq T$:

$$(7.9) \qquad \frac{d}{dt} \int u^p dv + \int \left| \nabla \left(u^{p/2} \right) \right|^2 \, dv \leq C \, p^{\frac{2n}{q-n}} \int u^p \, dv$$

where $C = C(n, q, p_0, C_S)$.

To apply Lemma 7.5, we need first to gain control of the Sobolev constant C_S defined in (7.7). To this end, let $\widehat{T}_2 \leq T_0$ denote the first time at which

$$(7.10) \qquad \int R_h^s \, dv_h = 2 \int R_g^s \, dv_g,$$

(if (7.10) never occurs then take $\widehat{T}_2 = +\infty$) and define $T_2 = \min\left(\widehat{T}_2, 1\right)$. By the definition of the Yamabe invariant, for $t \leq T_0$ and any $\varphi \in W^{1,2}(M^4)$,

$$(7.11) \qquad Y(g_0) \left(\int \varphi^4 \, dv_h \right)^{\frac{1}{2}} \leq \int 6 |\nabla \varphi|^2 \, dv_h + \int R_h \varphi^2 dv_h.$$

Using (7.10) and the fact that $s > 2$ we can estimate the second term on the right in (7.11) as follows:

$$\int R_h \, \varphi^2 \, dv_h \leq \left(\int R_h^s \, dv_h \right)^{\frac{1}{s}} \left(\int |\varphi|^{\frac{2s}{s-1}} dv_h \right)^{\frac{s-1}{s}}$$

$$\leq \left(2 \int R_g^s \, dv_g \right)^{\frac{1}{s}} \left(\int \varphi^4 \, dv_h \right)^{\frac{1}{s}} \left(\int \varphi^2 \, dv_h \right)^{\frac{s-2}{s}}.$$

By (7.10) we therefore have

$$\int R_h \, \varphi^2 \, dv_h \leq C(g_0) \left(\int \varphi^4 \, dv_h \right)^{\frac{1}{s}} \left(\int \varphi^2 \, dv_h \right)^{\frac{s-2}{s}}$$

$$\leq \frac{1}{2} Y(g_0) \left(\int \varphi^4 \, dv_h \right)^{\frac{1}{2}} + C \int \varphi^2 \, dv_h.$$

Substituting this into (7.11), we see that (7.7) holds for $0 \leq t \leq T_2$, with $C_S = C_S(g_0)$.

We now invoke Lemma 7.5 with $u = R$, $b = R$, $q = 2s > 8$, and $p_0 = s$. By (7.9),

$$(7.12) \qquad \frac{d}{dt} \int R^s \, dv \leq C \int R^s$$

where $C = C(g_0, s)$. Integrating (7.12) we obtain

$$(7.13) \qquad \int R_h^s \, dv_h \leq e^{Ct} \int R_g^s \, dv_g.$$

If $T_2 < 1$, then taking $t = T_2$ in (7.13) we conclude that

$$(7.14) \qquad T_2 = \frac{1}{C} \log 2 \geq C(g_0).$$

Also, from (7.8) we see that

$$(7.15) \qquad R \leq Ct^{-\frac{2}{s}}$$

for $0 \leq t \leq T_2$, with $C = C(g_0)$. Writing $h = e^{2v} g$ and differentiating, we see that v satisfies

$$(7.16) \qquad \begin{cases} \frac{\partial v}{\partial t} = -\frac{1}{6} R, \\ v(0, \cdot) = 0. \end{cases}$$

Integrating (7.16) using (7.15) we find

$$(7.17) \qquad \| v \|_\infty \leq C T_2^{1 - \frac{2}{s}} \leq C.$$

Now let $\widehat{T}_1 \leq T_0$ denote the first time at which

$$(7.18) \qquad \int |\mathrm{Ric}_h|^s \, dv_h = 2 \int |\mathrm{Ric}_g|^s \, dv_g,$$

(if (7.18) never occurs take $\widehat{T}_1 = +\infty$) and define $T_1 = \min\left\{\widehat{T}_1, T_2\right\}$. In order to apply Lemma 7.5 to the evolution for the Ricci tensor, we first derive the evolution equation for $|\mathrm{Ric}|^2$.

LEMMA 7.6.   *Under* (7.1),

$$(7.19) \quad \frac{\partial}{\partial t} |\mathrm{Ric}|^2 = \Delta |\mathrm{Ric}|^2 - 2|\nabla \mathrm{Ric}|^2 - 4\mathrm{tr}\, Ric^3 + 3R|\mathrm{Ric}|^2$$

$$- \frac{1}{3} R^3 + 4W_{ikj\ell} R_{ij} R_{k\ell} + 4B_{ij} R_{ij}$$

where B_{ij} is the Bach tensor.

   Proof. A simple calculation gives

$$\frac{\partial R_{ij}}{\partial t} = \frac{1}{3} \nabla_i \nabla_j R + \frac{1}{6} \Delta R \, g_{ij}.$$

Therefore, by (1.18),

$$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} - 2R_{ik} R_{jk} + \frac{1}{2} |\text{Ric}|^2 g_{ij} + \frac{2}{3} R R_{ij}$$
$$- \frac{1}{6} R^2 g_{ij} + 2W_{ikj\ell} R_{k\ell} + 2B_{ij}$$

and (7.19) follows.   □

COROLLARY 7.7.   *For $t \leq T_1$,*

(7.20)
$$\frac{\partial}{\partial t} |\text{Ric}| \leq \Delta |\text{Ric}| + C |\text{Ric}|^2.$$

Proof. By (7.19),

$$\frac{\partial}{\partial t} |\text{Ric}|^2 \leq \Delta |\text{Ric}|^2 - 2|\nabla \text{Ric}|^2 + C |\text{Ric}|^3$$
$$+ 4|W| |\text{Ric}|^2 + 4 |B| |\text{Ric}|.$$

Since $|W| = |W_h| = e^{-2v}|W_g|$, $|B| = |B_h| = e^{-4v}|B_g|$, by (7.17) it follows that for $t \leq T_1$,

(7.21) $$\frac{\partial}{\partial t} |\text{Ric}|^2 \leq \Delta |\text{Ric}|^2 - 2|\nabla \text{Ric}|^2 + C |\text{Ric}|^3 + C |\text{Ric}|^2 + C |\text{Ric}|$$
$$\leq \Delta |\text{Ric}|^2 - 2|\nabla \text{Ric}|^2 + C |\text{Ric}|^3 + C |\text{Ric}|.$$

By (7.4), $C_0 < R \lesssim |\text{Ric}|$, so $|\text{Ric}| \lesssim |\text{Ric}|^2$. Applying this inequality to (7.21) we obtain (7.20).   □

As a consequence of (7.20), we may apply Lemma 7.5 with $u = |\text{Ric}|$, $b = C|\text{Ric}|$, $q = 2s > 8$, and $p_0 = s$. By (7.9),

(7.22)
$$\frac{d}{dt} \int |\text{Ric}|^s dv \leq C \int |\text{Ric}|^s \, dv.$$

Integrating (7.22) in time gives

(7.23)
$$\int |\text{Ric}|^s \, dv \leq e^{Ct} \int |\text{Ric}_g|^s \, dv_g.$$

Now, if $T_1 < \min\{1, T_2\}$, then taking $t = T_1$ in (7.23) we see that $T_1 \geq C(\int |\text{Ric}_g|^s \, dv_g) = C(g_0)$. On the other hand, if $T_1 \geq \min\{1, T_2\}$, then by (7.14) we still conclude that $T_1 \geq C(g_0)$.

Finally, note that (7.8) implies part (ii) of Proposition 7.4, thus completing the proof.   □

PROPOSITION 7.8. *Define $f = \sigma_2(A) + 2\gamma_1 |\eta|^2$. If $t \leq T_1$, then under (7.1)*

(7.24) $$\frac{\partial}{\partial t} \left(\frac{f}{R} \right) \geq \Delta \left(\frac{f}{R} \right) + \frac{2}{R} \operatorname{tr} E^3 + \frac{1}{3} |E|^2 - \frac{1}{3} f$$
$$- 2R^{-1} W_{ikj\ell} E_{ij} E_{k\ell} - 2R^{-1} B_{ij} E_{ij} - C.$$

Proof. The proof of (7.24) requires several intermediate lemmas, beginning with

LEMMA 7.9.   *Under* (7.1),

$$
(7.25) \quad \frac{\partial f}{\partial t} = \Delta f + \left(|\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right) + 2\mathrm{tr}\, E^3 + \frac{1}{3}\, R|E|^2
$$

$$
- 2W_{ikj\ell}\, E_{k\ell}\, E_{ij} - 2B_{ij}\, E_{ij} + \left(\frac{4}{3}\, \gamma_1\, R|\eta|^2 - 2\gamma_1\, \Delta\, |\eta|^2 \right).
$$

Proof. Since

$$
\frac{\partial}{\partial t} |\eta|^2 = \frac{2}{3}\, R\, |\eta|^2,
$$

by combining (7.3) and (7.19) we get (7.25).

LEMMA 7.10.

$$
(7.26) \quad |\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \geq -2\left\langle \nabla f, \frac{\nabla R}{R} \right\rangle + 2f\, \frac{|\nabla R|^2}{R^2} + 4\,\gamma_1\, |\nabla|\eta|\, |^2.
$$

Proof. We argue as we did in Lemma 5.6. Namely,

$$
\begin{aligned}
\nabla f &= \nabla(\sigma_2(A) + 2\gamma_1|\eta|^2) \\
&= \nabla\left(-\frac{1}{2} |E|^2 + \frac{1}{24}\, R^2 + 2\gamma_1|\eta|^2 \right) \\
&= -|E|\nabla|E| + \frac{1}{12}\, R\nabla R + 4\gamma_1|\eta|\nabla|\eta|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
-\left\langle \nabla f, \frac{\nabla R}{R} \right\rangle &= \frac{|E|}{R} \left\langle \nabla|E|, \nabla R \right\rangle - \frac{1}{12} |\nabla R|^2 - 4\gamma_1 \frac{|\eta|}{R} \left\langle \nabla|\eta|, \nabla R \right\rangle \\
&\leq \frac{1}{2}|\nabla E|^2 + \frac{1}{2}\frac{|E|^2}{R^2} |\nabla R|^2 - \frac{1}{12}|\nabla R|^2 \\
&\quad - 4\gamma_1 \frac{|\eta|}{R} \left\langle \nabla|\eta|, \nabla R \right\rangle \\
&= \frac{1}{2}|\nabla E|^2 + \frac{|\nabla R|^2}{R^2} \left[-f + \frac{1}{24}\, R^2 + 2\gamma_1|\eta|^2 \right] \\
&\quad - \frac{1}{12} |\nabla R|^2 - 4\gamma_1 \frac{|\eta|}{R} \left\langle \nabla|\eta|, \nabla R \right\rangle \\
&= \frac{1}{2} \left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2 \right) - f\, \frac{|\nabla R|^2}{R^2} \\
&\quad + 2\gamma_1\, |\eta|^2\, \frac{|\nabla R|^2}{R^2} - 4\gamma_1 \frac{|\eta|}{R} \left\langle \nabla|\eta|, \nabla R \right\rangle
\end{aligned}
$$

$$\leq \ \frac{1}{2}\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) - f\frac{|\nabla R|^2}{R^2}$$

$$+ 2\gamma_1|\eta|^2\frac{|\nabla R|^2}{R^2} - 2\gamma_1|\nabla|\eta||^2$$

$$= \ \frac{1}{2}\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) - f\frac{|\nabla R|^2}{R^2} - 2\gamma_1|\nabla|\eta||^2. \quad \square$$

LEMMA 7.11.   *For $t \leq T_1$,*

$$(7.27) \qquad \frac{4}{3}\gamma_1 R|\eta|^2 - 2\gamma_1\Delta|\eta|^2 + 4\gamma_1|\nabla|\eta||^2 \geq -CR - C.$$

Proof. Let us write $h = e^{2v}g = e^{2(v+w)}g_0 \equiv e^{2z}g_0$. Let $L = \Delta - \frac{1}{6}R$ denote the conformal Laplacian; then $L_h\varphi = e^{-3z}L_{g_0}(e^z\varphi) = e^{-3z}L_0(e^z\varphi)$. Therefore,

$$(7.28) \quad \Delta|\eta|^2 \ = \ L_h|\eta|^2 + \frac{1}{6}R|\eta|^2$$

$$\geq \ e^{-3z}L_0(e^z|\eta|^2)$$

$$= \ e^{-3z}L_0\left(e^{-3z}|\eta|_0^2\right)$$

$$= \ e^{-3z}\left(\Delta_0\left(e^{-3z}|\eta|_0^2\right) - \frac{1}{6}R_0e^{-3z}|\eta|_0^2\right]$$

$$= \ e^{-3z}\left[|\eta|_0^2\Delta_0\left(e^{-3z}\right) + e^{-3z}\Delta_0|\eta|_0^2\right.$$

$$\left. + 2\left\langle\nabla_0\left(e^{-3z}\right), \nabla_0\left(|\eta|_0^2\right)\right\rangle - \frac{1}{6}R_0e^{-3z}|\eta|_0^2\right]$$

$$= \ e^{-6z}|\eta|_0^2\left[-3\Delta_0z + 9|\nabla_0z|^2\right]$$

$$+ e^{-6z}\Delta_0|\eta|_0^2 - 12e^{-6z}\langle\nabla_0z, \nabla_0|\eta|_0\rangle|\eta|_0$$

$$- \frac{1}{6}R_0e^{-3z}|\eta|_0^2.$$

By (1.10),

$$\Delta_0z + |\nabla_0z|^2 + \frac{1}{6}Re^{2z} = \frac{1}{6}R_0.$$

Thus,

$$e^{-6z}|\eta|_0^2\left[-3\Delta_0z + 9|\nabla_0z|^2\right] = e^{-6z}|\eta|_0^2\left[12|\nabla_0z|^2 + \frac{1}{2}Re^{2z} - \frac{1}{2}R_0\right].$$

Substituting this into (7.28), we get

$$\Delta|\eta|^2 \ \geq \ 12\,e^{-6z}|\eta|_0^2|\nabla_0z|^2 - 12\,e^{-6z}\langle\nabla_0z, \nabla_0|\eta|_0\rangle|\eta|_0$$

$$+ e^{-6z}\Delta_0|\eta|_0^2 - \frac{2}{3}R_0e^{-6z}|\eta|_0^2.$$

Since $\gamma_1 < 0$, this implies

$$
\begin{aligned}
-2\gamma_1\Delta|\eta|^2 \geq \ & -24\gamma_1\, e^{-6z}\, |\eta|_0^2\, |\nabla_0 z|^2\ +\ 24\gamma_1\, e^{-6z}\, \langle\nabla_0 z, \nabla_0|\eta|_0\rangle\, |\eta|_0 \\
& -2\gamma_1\, e^{-6z}\, \Delta_0|\eta|_0^2\ +\ \frac{4}{3}\,\gamma_1\, R_0 e^{-6z}|\eta|_0^2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
4\gamma_1|\nabla|\eta|\,|^2\ &=\ 4\gamma_1 e^{-6z}|\nabla_0(e^{-2z}|\eta|_0)|^2 \\
&=\ 4\gamma_1 e^{-6z}|-2e^{-2z}\,\nabla_0 z\,|\eta|_0\ +\ e^{-2z}\,\nabla_0|\eta|_0|^2 \\
&=\ 16\gamma_1\, e^{-6z}|\nabla_0 z|^2\, |\eta|_0^2\ -\ 16\gamma_1\, e^{-6z}\,\langle\nabla_0 z, \nabla_0|\eta|_0\rangle\, |\eta|_0 \\
&\quad +4\gamma_1 e^{-6z}|\nabla_0|\eta|_0|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
-2\gamma_1\Delta|\eta|^2\ &+\ 4\gamma_1|\nabla|\eta|\,|^2 \\
\geq\ & -8\gamma_1\, e^{-6z}|\eta|_0^2\,|\nabla_0 z|^2 \\
& +8\gamma_1 e^{-6z}\,\langle\nabla_0 z, \nabla_0|\eta|_0\rangle|\eta|_0 \\
& +4\gamma_1 e^{-6z}|\nabla_0|\eta|_0|^2\ -\ 2\gamma_1\, e^{-6z}\Delta_0|\eta|_0^2\ +\ \frac{4}{3}\,\gamma_1 R_0 e^{-6z}|\eta|_0^2 \\
\geq\ & 6\gamma_1 e^{-6z}|\nabla_0|\eta|_0|^2\ -\ 2\gamma_1\, e^{-6z}\Delta_0|\eta|_0^2\ +\ \frac{4}{3}\,\gamma_1\, R_0\, e^{-6z}|\eta|_0^2.
\end{aligned}
$$

By Proposition 7.4 and (5.3), $\|\,z\,\|_\infty \leq \|\,w\,\|_\infty\ +\ \|\,v\,\|_\infty \leq C(g_0)$. Hence,

$$
-2\gamma_1\,\Delta|\eta|^2\ +\ 4\gamma_1|\nabla|\eta|\,|^2\ \geq\ -C.
$$

Also,

$$
\frac{4}{3}\,\gamma_1\, R|\eta|^2 = \frac{4}{3}\,\gamma_1\, Re^{-4z}\,|\eta|_0^2 \geq -CR,
$$

so that (7.27) follows. $\square$

To complete the proof of (7.24), we compute $\frac{\partial}{\partial t}\left(\frac{f}{R}\right)$ using the results of Lemmas 7.9 and 7.10:

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{f}{R}\right)\ &=\ R^{-1}\frac{\partial f}{\partial t}\ +\ f\frac{\partial}{\partial t}\left(R^{-1}\right) \\[2mm]
&=\ R^{-1}\frac{\partial f}{\partial t}\ -\ R^{-2}f\left(\Delta R + \frac{1}{3}R^2\right) \\[2mm]
&\geq\ R^{-1}\Delta f\ -\ 2R^{-2}\,\langle\nabla f, \nabla R\rangle\ +\ 2f\,R^{-3}|\nabla R|^2 \\[2mm]
&\quad -\ f\,R^{-2}\Delta R\ +\ 2R^{-1}\,\mathrm{tr}\,E^3\ +\ \frac{1}{3}\,|E|^2\ -\ \frac{1}{3}f \\[2mm]
&\quad -\ 2R^{-1}W_{ikj\ell}\,E_{ij}\,E_{k\ell}\ -\ 2R^{-1}\,B_{ij}\,E_{ij}\ -\ CR^{-1}\ -\ C.
\end{aligned}
$$

Note that

$$\Delta\left(\frac{f}{R}\right) = R^{-1}\Delta f + f\Delta(R^{-1}) + 2\langle\nabla f, \nabla(R^{-1})\rangle$$

$$= R^{-1}\Delta f - fR^{-2}\Delta R + 2fR^{-3}|\nabla R|^2 - 2R^{-2}\langle\nabla f, \nabla R\rangle,$$

so that (7.24) follows.   □

PROPOSITION 7.12.   *Define* $\varphi = \max\left\{-\frac{f}{R}, 0\right\}$. *Then for* $t \leq T_1$,

(7.29) $$\frac{\partial\varphi}{\partial t} \leq \Delta\varphi + C_1|\mathrm{Ric}|\varphi + C_1|\mathrm{Ric}|$$

where $C_1 = C_1(g_0)$.

Proof. We begin by analyzing the curvature terms in (7.24). As in the proof of Corollary 5.8, we have the sharp inequality

$$2R^{-1}\mathrm{tr}\,E^3 + \frac{1}{3}|E|^2 = \frac{1}{3R}\left(6\mathrm{tr}\,E^3 + R|E|^2\right)$$

$$\geq \frac{|E|^2}{3R}\left(-2\sqrt{3}|E| + R\right).$$

Therefore,

(7.30)

$$2R^{-1}\mathrm{tr}\,E^3 + \frac{1}{3}|E|^2 \geq \frac{|E|^2}{3R(2\sqrt{3}|E| + R)}(2\sqrt{3}|E| + R)(-2\sqrt{3}|E| + R)$$

$$= \frac{|E|^2}{3R(2\sqrt{3}|E| + R)}(-12|E|^2 + R^2)$$

$$= \frac{8|E|^2}{R(2\sqrt{3}|E| + R)}\sigma_2(A)$$

$$= \frac{8|E|^2}{R(2\sqrt{3}|E| + R)}(f - 2\gamma_1|\eta|^2)$$

$$\geq \frac{8|E|^2}{(2\sqrt{3}|E| + R)}\left(\frac{f}{R}\right).$$

Also, by Proposition 7.4 (iii),

(7.31) $$-2R^{-1}W_{ikj\ell}E_{ij}E_{k\ell} - 2R^{-1}B_{ij}E_{ij}$$

$$\geq -C\frac{|E|^2}{R} - C\frac{|E|}{R}$$

$$\geq -C\frac{|E|^2}{R} - C\frac{1}{R}$$

$$= \frac{C}{R}\left(-\frac{1}{2}|E|^2 + \frac{1}{24}R^2\right) - \frac{C}{24}R - \frac{C}{R}$$

$$\begin{aligned}
&= \frac{C}{R}\left(f - 2\gamma_1|\eta|^2\right) - \frac{C}{24}R - \frac{C}{R} \\
&\geq C\frac{f}{R} - CR - \frac{C}{R}.
\end{aligned}$$

Finally, combining (7.24), (7.30), (7.31), and using (7.4) we conclude

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{f}{R}\right) \geq {}& \Delta\left(\frac{f}{R}\right) + \frac{8|E|^2}{(2\sqrt{3}|E|+R)}\left(\frac{f}{R}\right) - \frac{1}{3}R\left(\frac{f}{R}\right) \\
& - C\left(\frac{f}{R}\right) - CR - C.
\end{aligned}$$

If we let $\varphi = \max\{-\frac{f}{R}, 0\}$, then (in the $W^{1,2}$-sense)

$$\frac{\partial\varphi}{\partial t} \leq \Delta\varphi + C(|E| + R + 1)\varphi + CR + C.$$

Since $|\text{Ric}| \geq \frac{1}{2}R \geq C > 0$, $|E| + R + 1 \lesssim |\text{Ric}|$, and (7.29) follows. $\qquad\square$

Proof of Theorem 7.1.   Let us begin by summarizing (7.29) and Proposition 7.4(ii): for $t \leq T_1$,

$$(7.32) \qquad\qquad \frac{\partial\varphi}{\partial t} \leq \Delta\varphi + C_1|\text{Ric}|\varphi + C_1|\text{Ric}|,$$

$$(7.33) \qquad\qquad \|\text{ Ric }\|_\infty \leq C_2 t^{-\frac{2}{s}}.$$

Define $\varphi_1 = \varphi_1(t) = \exp\left\{\frac{s}{s-2}C_1C_2 t^{\frac{s-2}{s}}\right\} - 1$. Since $s > 4$, $\varphi_1(0) = 0$ and it is easily verified that $\partial_t\varphi_1 = C_1C_2(1 + \varphi_1)t^{-\frac{2}{s}}$ for $t > 0$. Let $u = \varphi - \varphi_1$. Then

$$\begin{aligned}
\frac{\partial u}{\partial t} ={}& \frac{\partial\varphi}{\partial t} - \frac{\partial\varphi_1}{\partial t} \\
\leq{}& \Delta\varphi + C_1|\text{Ric}|\varphi + C_1|\text{Ric}| - \frac{\partial\varphi_1}{\partial t} \\
={}& \Delta u + C_1|\text{Ric}|u + C_1|\text{Ric}|\varphi_1 + C_1|\text{Ric}| - \frac{\partial\varphi_1}{\partial t} \\
\leq{}& \Delta u + C_1|\text{Ric}|u + C_1C_2(1 + \varphi_1)t^{-\frac{2}{s}} - \frac{\partial\varphi_1}{\partial t} \\
={}& \Delta u + C_1|\text{Ric}|u.
\end{aligned}$$

Appealing once more to Lemma 7.5 with $b = C_1|\text{Ric}|$, $p_0 = 2$, $q = 2s$, we conclude that for $t \leq T_1$,

$$\|\varphi - \varphi_1\|_\infty \leq Ct^{-1}\|\varphi(0, \cdot) - \varphi_1(0)\|_{L^2}.$$

Now, $\varphi_1(0) = 0$, and by $(*)_\delta$

$$\|\varphi(0, \cdot)\|_{L^2} \leq \|\sigma_2(A) + 2\gamma_1|\eta|^2\|_{L^2} = \left\|\frac{\delta}{4}\left(\frac{\Delta_g R_g}{R_g}\right)\right\|_{L^2}.$$

Therefore, by Corollary 5.25,

$$\| \varphi - \varphi_1 \|_\infty \leq C\delta^{\frac{1}{2}} t^{-1}$$

$$\Longrightarrow \varphi \leq \varphi_1(t) + C\delta^{\frac{1}{2}} t^{-1}, \quad t \leq T_1.$$

By the definition of φ, this implies

$$\frac{1}{R} \left(\sigma_2(A) + 2\gamma_1|\eta|^2 \right) \geq -\varphi_1(t) - C\delta^{\frac{1}{2}} t^{-1}, \quad t \leq T_1.$$

By Taylor's Theorem, $\varphi_1(t) \leq Ct^{1-\frac{2}{s}}$. Also, by (7.33), $R \leq Ct^{-\frac{2}{s}}$. Therefore,

$$\sigma_2(A) + 2\gamma_1|\eta|^2 \geq R\left(-\varphi_1(t) - C\delta^{\frac{1}{2}}t^{-1} \right) \geq Ct^{-\frac{2}{s}}\left(-t^{1-\frac{2}{s}} - \delta^{\frac{1}{2}}t^{-1} \right)$$

$$\Longrightarrow \sigma_2(A) \geq -2\gamma_1|\eta|^2 - C_3\, t^{1-\frac{4}{s}} - C_3\, \delta^{\frac{1}{2}}\, t^{-(1+\frac{2}{s})}, \quad t \leq T_1.$$

Recall that $|\eta|^2 = e^{-4(v+w)}|\eta|_0^2 \geq C(g_0) > 0$. Thus, there is a constant $C_4 = C_4(g_0) > 0$ such that

(7.34) $$\sigma_2(A) \geq C_4 - C_3\, t^{1-\frac{4}{s}} - C_3\, \delta^{\frac{1}{2}}\, t^{-(1+\frac{2}{s})}, \quad t \leq T_1.$$

Let $\widehat{t}_0 = \widehat{t}_0(g_0)$ satisfy

$$C_3\, \widehat{t}_0^{\,1-\frac{4}{s}} = \frac{1}{4}\, C_4$$

and define $t_0 = \min\left\{ T_1, \widehat{t}_0 \right\}$. Then t_0 satisfies

(7.35) $$C_3\, t_0^{1-\frac{4}{s}} \leq \frac{1}{4}\, C_4$$

because $s > 4$.

It now follows from (7.34) and (7.35) that the metric $h = h(t_0, \cdot)$ satisfies

$$\sigma_2(A_h) \geq \frac{3}{4}\, C_4 - C_3\, \delta^{\frac{1}{2}}\, t_0^{-(1+\frac{2}{s})}.$$

Therefore, once $\delta < \delta_0 = \left(C_4 t_0^{1+\frac{2}{s}} \Big/ 4C_3 \right)^2$,

$$\sigma_2(A_h) \geq \frac{1}{2}\, C_4 > 0. \qquad \square$$

8. Examples

In this section we consider the class of 4-manifolds that admits a positive conformal structure (i.e., the Yamabe constant of the conformal class is positive) satisfying the condition $\int \sigma_2(A)dv > 0$. As a consequence of our result these manifolds carry metrics of positive Ricci curvature, hence their fundamental group must be finite. We consider simply connected manifolds

satisfying these two conditions. The homeomorphism classification of simply connected 4-manifolds proceeds according to the algebraic classification of the intersection form (see [DK]). There are two families of possible quadratic forms according to whether the manifold carries a spin structure. For the nonspin case, the quadratic form is of odd type and is of the form

$$(8.1) \qquad \underbrace{1 \oplus 1 \oplus \cdots \oplus 1}_{k} \oplus \underbrace{-1 \oplus \cdots \oplus -1}_{\ell}$$

where $k \neq \ell$. By reversing orientation if necessary we may assume $k > \ell$. M is then homeomorphic to $k(\mathbb{CP}^2) \# \ell \left(\overline{\mathbb{CP}^2}\right)$. The index formula

$$12\pi^2 \tau(M) = \frac{1}{4} \int \left(|W^+|^2 - |W^-|^2 \right)$$

and the Gauss Bonnet formula

$$8\pi^2 \chi(M) = \frac{1}{4} \int \left(|W^+|^2 + |W^-|^2 \right) + \int \sigma_2$$

combine to give

$$(8.2) \qquad \begin{cases} 4\pi^2(2\chi + 3\tau) = \frac{1}{2} \int |W^+|^2 + \int \sigma_2, \\ 4\pi^2(2\chi - 3\tau) = \frac{1}{2} \int |W^-|^2 + \int \sigma_2. \end{cases}$$

Since $\chi(M) = k + \ell + 2$, $\tau(M) = k - \ell$. In order that $\int_M \sigma_2 > 0$ for $M = k(\mathbb{CP}^2) \# \ell(\overline{\mathbb{CP}^2})$, it is necessary that $4 + 5\ell > k$. In particular, when $\ell = 0$, $k < 4$. In fact Lebrun et al. [LNN] show that $k(\mathbb{CP}^2)$ for $k \leq 3$ admits a conformal structure which is self-dual and satisfies $\frac{R}{2\sqrt{3}} > |E|$. When $\ell = 1$, $k < 9$, and each such topology is homeomorphic to a Kähler metric with positive first Chern class. When $\ell > 1$, the constraint $2\chi - 3\tau = 4 - k + 5\ell > 0$ is satisfied by a large number of manifolds; however it is not clear whether all such topologies admit a positive conformal class for which $\int \sigma_2 dv > 0$.

In case M is spin, the quadratic form is of even-type and hence of the form

$$(8.3) \qquad k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2\ell \, E_8$$

where E_8 is the matrix corresponding to the Dynkin diagram of the exceptional Lie group E_8. The vanishing theorem of Lichnerowicz requires the $\hat{A}$ genus to vanish; hence the signature is zero. This means no E_8 component appears in the quadratic form, so that M is homeomorphic to $k(S^2 \times S^2)$.

To summarize the above discussion: the simply connected 4-manifolds that admit a positive conformal structure with $\int \sigma_2 dv > 0$ must be homeomorphic to $k(\mathbb{CP}^2) \# \ell \left(\overline{\mathbb{CP}^2}\right)$ or $k(S^2 \times S^2)$.

We remark that Sha-Yang [ShYa] have constructed metrics of positive Ricci curvature on $k(\mathbb{CP}^2) \# \ell(\overline{\mathbb{CP}^2})$ and $k(S^2 \times S^2)$ without constraints on

k, ℓ. Thus, the class of 4-manifolds admitting metrics with positive σ_2 are necessarily a proper subset of those admitting positive Ricci curvature metrics.

In the following we give a construction:

PROPOSITION 8.1. *Given two positive conformal structures* (M', g') *and* (M'', g''), *there exists a positive conformal structure on* $M_1 \# M_2$ *so that*

$$\int_{M'} |W_+|^2 + \int_{M''} |W_+|^2 - \int_{M' \# M''} |W_+|^2$$

is arbitrarily small.

Remark. This fact is probably well known in view of the work of Schoen-Yau [ScY] and Gromov-Lawson [GL]. However, we cannot find it in the literature and so will provide an argument.

Proof. Take a geodesic coordinate system B' centered at some $p' \in M'$ in which $g'_{ij}(x) = \delta_{ij} + G'_{imjn} x_m x_n + O(|x|^3)$ and likewise for B'', p'' and $g''_{ij} = \delta_{ij} + G''_{imjn} x_m x_n + O(|x|^3)$. Subject $B' \setminus \{p'\}$ to the conformal change of metric

(8.4)
$$\tilde{g}' = |x|^{-2} g'_{ij} dx^i dx^j = dt^2 + \left(\delta_{ij} + G'_{ijmn} \sigma_m \sigma_n e^{2t} + O(e^{3t}) \right) d\sigma^i d\sigma^j,$$

where $t = \log|x|$, $\sigma = \frac{x}{|x|}$. To compute the scalar curvature of $\tilde{g}'$, we notice that

(8.5)
$$\begin{aligned}
-6\Delta(|x|^{-1}) + R|x|^{-2} &= (-6\partial_r^2 - 6H\partial_r)|x|^{-1} + R|x|^{-1} \\
&= -12|x|^{-3} + 6\left(O(|x|) + \frac{3}{|x|} \right) |x|^{-2} + R|x|^{-1} \\
&= (6 + O(|x|))|x|^{-3},
\end{aligned}$$

where H is the mean curvature. So the scalar curvature is

(8.6)
$$\tilde{R} = 6 + O(|x|).$$

On the annuli $\{e^{-r} < |x| < e^{-s}\}$ and $\{e^{-r} < |x'| < e^{-s}\}$, introduce the cylindrical coordinates $-r < t < -s$ and $\sigma \in S^3$ for B_1 and $-r < t' < -s$ and $\sigma' \in S^3$ for B'_1; where $t = \log|x|$, $t' = \log|x'|$, $\sigma = \frac{x}{|x|}$, $\sigma' = \frac{x'}{|x'|}$. We make the identification $(t, \sigma) \sim (t', \sigma')$ if $t + r = -(s + t')$ and $\sigma = \sigma'$, to form the connected sum $(B_1 \setminus \{|x| < e^{-r}\}) \cup (B'_1 \setminus \{|x'| < e^{-r}\})$. Let ρ be a smooth function on $[0, 1]$, $\rho(0) = 0$ and $\rho(1) = 1$, $\rho'(0) = \rho'(1) = 0$. Then set $\varphi(t) = \rho\left(\frac{t+r}{r-s}\right)$ and define the gluing metric

$$
\begin{aligned}
h_{ij}\, dx^i\, dx^j &= (\varphi(t)\tilde{g}'_{ij}\,(t,\sigma) + (1-\varphi)(t)\,\tilde{g}''_{ij}\,(t',\sigma))dx^i\, dx^j \\
&= dt^2 + (\delta_{ij} + (\varphi G'_{ijmn}\sigma_m\sigma_n + (1-\varphi)\,G''_{ijmn}\sigma_m\sigma_n)\,e^{2t} \\
&\quad + O(e^{3t}))\, d\sigma^i\, d\sigma^j.
\end{aligned}
$$

Over the annuli $\{e^{-s} < |x| < 1\}$ and $\{e^{-s} < |x'| < 1\}$ we introduce the conformal metric $g = e^{2t+v(t)}\tilde{g}'$ to smoothly join the metric g' to $\tilde{g}'$ (respectively g'' to $\tilde{g}''$), while keeping the scalar curvature positive.

Observe that for functions f of the t variable we have $\tilde{\Delta}' = \partial_t^2 + \tilde{H}'\partial_t$, where $\tilde{H}'$ is the mean curvature of the t slice $\{t\} \times S^3$. The scalar curvature R_g is then given by

$$
\begin{aligned}
R_g e^{3(t+v)} &= -6\Delta_{\tilde{g}'} e^{t+v} + R_{\tilde{g}'} e^{t+v} \\
&= R_{g'} e^{3t} e^v - 12 e^{t+v}\partial_t v - 6\left(v'' + \tilde{H}'v' + |v'|^2\right) e^{t+v} \\
&= \left\{R_{g'} e^{2t} - 12v' - 6\left(v'' + |v'|^2 + o(|v'|)\right)\right\} e^{t+v}.
\end{aligned}
$$

It is thus clear that if s is taken sufficiently large, we can find a function v defined on $[-s, 0]$ which agrees with the constant zero function to second order at $t = 0$ and agrees with the linear function $-t$ up to second order at $t = -s$, while keeping the scalar curvature positive. Similarly we can join the metrics g'' to $\tilde{g}''$ over the annulus $\{e^{-s} < |x'| < 1\}$. For the gluing region, write

$$
(8.7) \qquad h_{ij} = (h_0)_{ij} + e^{2t}\left(\varphi G'_{imjn} + (1-\varphi)\,G''_{imjn}\right)\sigma_m\sigma_n + O(e^{3t})
$$

where $(h_0) = $ cylinder metric. In computing Γ^m_{ij} we will write

$$
\Gamma^m_{ij} = \frac{1}{2}\, h^{mn}\,(h_{nj,i} + h_{in,j} - h_{ij,n})
$$

and will replace indices with dots to write in abbreviation. Thus for example $G'_{..}$ will mean $G'_{imjn}\sigma_m\sigma_n$.

$$
\begin{aligned}
\Gamma_{..} &= \frac{1}{2}\, h^{..}\,(\partial.\, h_{..}) \\
&= \frac{1}{2}\, h^{..}(\partial.(h_0)_{..} + (\varphi\partial.G'_{..} + (1-\varphi)\,\partial.G''_{..})e^{2t} + \partial.\varphi(G_{..})e^{2t} \\
&\quad + \text{(lower order terms)}.
\end{aligned}
$$

Observe that when the differentiation falls on φ, $|\nabla.\varphi| = O\left(\frac{1}{r-s}\right)$; hence for $r - s$ large it is of lower order. Thus we write

$$
\Gamma_{..} = (\Gamma_0)_{..} + \frac{1}{2}\, h^{..}\left[\varphi\partial.(G'_{..}e^{2t}) + (1-\varphi)\,\partial.(G''_{..}e^{2t}) + O(e^{3t})\right].
$$

Hence

$$
(8.8) \qquad \Gamma_{..} = (\Gamma_0)_{..} + O(e^{2t}) + O\left(\frac{1}{r-s}\right) e^{2t} + O(e^{3t}).
$$

Likewise

$$(8.9) \quad \Gamma_{..}^{.} \Gamma_{..}^{.} = \frac{1}{2} h^{..} \left\{ \partial. h_{0..} + \partial. (e^{2t} G'_{..}) \varphi + \partial. (e^{2t} G''_{..}) (1 - \varphi) + (\text{l.o.t.}) \right\}$$

$$\frac{1}{2} h^{..} \left\{ \partial. h_{0..} + \partial. (e^{2t} G'_{..}) \varphi + \partial. (e^{2t} G''_{..}) (1 - \varphi) + (\text{l.o.t.}) \right\}$$

$$= (\Gamma_0)^{.}_{..} (\Gamma_0)^{.}_{..} + O(e^{2t}) + O\left(e^{2t} \frac{1}{r-s} \right) + (\text{l.o.t.}) .$$

Hence,

$$(8.10) \qquad R^n_{ijm} = \mathring{R}^n_{ijm} + O(e^{2t}) + O(e^{2t} \frac{1}{r-s}) + (\text{l.o.t.})$$

where $\mathring{R}^n_{ijm}$ is the curvature tensor of the cylinder. This shows

$$(8.11) \qquad\qquad\qquad R = R_0 + O(e^{2t})$$

and

$$(8.12) \qquad\qquad |W|^2 = |W_0|^2 + O(e^{2t}) = O(e^{2t}).$$

Upon integration, the length of the cylinder being $r - s$, we find

$$(8.13) \qquad\qquad \int_{-r < t < -s} |W^+|^2 \, dv \leq O(|r-s|) (e^{-4r}). \qquad\qquad \square$$

We can apply the gluing construction above to show that $M = 2(S^2 \times S^2)$ admits a conformal structure satisfying

$$\frac{1}{4} \int_M |W^+|^2 \leq \frac{64\pi^2}{3} + \varepsilon.$$

Hence

$$\frac{1}{4\pi^2} \int_M \sigma_2 = 12 - \frac{1}{8\pi^2} \int |W^+|^2 \geq 12 - \frac{32}{3} - \varepsilon > 0.$$

Remark. This calculation becomes critical at $3(S^2 \times S^2)$.

Similarly $\mathbb{CP}^2 \#2(S^2 \times S^2)$ admits a conformal structure satisfying

$$\frac{1}{4} \int |W^+|^2 \leq \frac{64\pi^2}{3} + 12\pi^2 + \varepsilon,$$

so that

$$\frac{1}{4\pi^2} \int \sigma_2 \geq 17 - \frac{50}{3} - \varepsilon > 0.$$

Remarks added in proof.

1. The regularity of general weak solutions to critical exponential variational equations such as Euler equations of the functional F_δ appearing in this paper for $\delta \neq 0$ has been established in the article of Uhlenbeck and

Viaclovsky ([UV]). Thus in the proof of Proposition 4.3, one may quote this regularity result to verify smoothness of solutions.

2. Since the submission of this paper, we were able to use the techniques of this paper to establish a conformally invariant sphere theorem for 4-manifolds with positive Yamabe invariant satisfying an equality involving the Euler number and the L^2 integral of the Weyl tensor. We will address this result in another publication.

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